**University of Alberta**

# Hall and Knight's Higher Algebra: The Theory of Equations

by

Thomas Holloway ©

A thesis submitted to the Faculty of Graduate Studies and Research in
partial fulfillment of the requirements for the degree of Master of Science

in Mathematics

Department of Mathematical and Statistical Sciences

Edmonton, Alberta
Fall 2003

National Library
of Canada

Acquisitions and
Bibliographic Services

395 Wellington Street
Ottawa ON K1A 0N4
Canada

Bibliothèque nationale
du Canada

Acquisisitons et
services bibliographiques

395, rue Wellington
Ottawa ON K1A 0N4
Canada

*Your file Votre référence*
*ISBN: 0-612-87782-5*
*Our file Notre référence*
*ISBN: 0-612-87782-5*

The author has granted a non-exclusive licence allowing the National Library of Canada to reproduce, loan, distribute or sell copies of this thesis in microform, paper or electronic formats.

The author retains ownership of the copyright in this thesis. Neither the thesis nor substantial extracts from it may be printed or otherwise reproduced without the author's permission.

L'auteur a accordé une licence non exclusive permettant à la Bibliothèque nationale du Canada de reproduire, prêter, distribuer ou vendre des copies de cette thèse sous la forme de microfiche/film, de reproduction sur papier ou sur format électronique.

L'auteur conserve la propriété du droit d'auteur qui protège cette thèse. Ni la thèse ni des extraits substantiels de celle-ci ne doivent être imprimés ou aturement reproduits sans son autorisation.

Canada

**University of Alberta**

**Library Release Form**

**Name of Author:** Thomas Holloway

**Title of Thesis:** Hall and Knight's Higher Algebra: The Theory of Equations

**Degree:** Master of Science

**Year this Degree Granted:** 2003

Permission is hereby granted to the University of Alberta to reproduce single copies of this thesis and to lend or sell such copies for private, scholarly or scientific research purposes only.

The author reserves all other publication and other rights in association with the copyright in the thesis, and except as herein before provided, neither the thesis nor any substantial portion thereof may be printed or otherwise reproduced in any material form whatever without the author's prior written permission.

Date: Sept. 30/2003

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Hall and Knight's Higher Algebra: The Theory of Equations** submitted by **Thomas Holloway** in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

Prof. Andy Liu (Supervisor)

Prof. James E. Lewis

Prof. Elaine Simmt

Date: Aug 27, 2003

# Abstract

In 2001 the Edmonton Public School Board in cooperation with the University of Alberta sponsored an initiative to encourage middle school teachers of mathematics to upgrade their mathematics knowledge. Several courses were developed to meet this need, among them MATH 164: Higher Algebra. The namesake and content of that course were provided by a classic textbook from 1887 by Hall and Knight. By assisting with the development of, attending the classes of, and grading the assignments for the course over two offerings this study was compiled. This thesis consists of the course notes as offered in Winter 2003 accompanied by comments and an improved version of the text which will be published for future use and as a tribute to Hall and Knight's original.

# Acknowledgement

I would like to thank the University of Alberta and the Government of Yukon for their financial support; Andy Liu for his encouragement, dedication, and puns; the students of MATH 164 for having me in their class; my graduate student colleages, especially Jung Min Lee, for their help and company; the beautiful Loretta; and my wonderful family.

## TABLE OF CONTENTS

Introduction ........ 1

Part 1 ........ 3
1. Complex numbers ........ 4
2. The Theory of Quadratic Equations ........ 13
3. The Values of Rational Expressions ........ 24
4. Miscellaneous Equations ........ 30
5. Inequalities ........ 38
6. The Theory of Equations ........ 48
7. The Roots of Equations ........ 55
8. Transformation of Equations ........ 63
9. Cubic and Quartic Equations ........ 72

Part 2 ........ 81
Preface ........ 82
0. Preliminaries ........ 84
1. Quadratic Equations ........ 93
2. Quadratic Equations with Imaginary Roots 95
3. Miscellaneous Equations ........ 111
4. Equations of Higher Order ........ 119
Interlude ........ 141
5. Quartic Equations ........ 144
6. Cubic Equations ........ 148
7. Inequalities ........ 157
8. Symmetric Identities ........ 172

# INTRODUCTION

## Hall and Knight's Higher Algebra

Higher Algebra is a classic math textbook that was first published in 1887 and has been used as a standard text all around the world. Hall and Knight collaborated for a number of texts published by MacMillan who report that 189,000 copies of Higher Algebra were sold between 1887 and 1935. The book went out of print in 1977, although it has been picked up by at least one other publisher in India.

Higher Algebra is a diverse opus of some thirty-five chapters. Although many people studied (and some still do study) the text for A-level mathematics the material is now typically distributed over several university courses.

Higher Algebra, a British text written in English, is an embodiment of mathematical and educational colonialism. As the mathematics education community attempts to "restore dignity to the losers" through ethno-mathematics research and education[1], it is interesting to analyse Hall and Knight's Higher Algebra; the face of western mathematics that went abroad to the kingdom on which the sun never set.

## The Thesis

This thesis consists of two parts. Part 1 is a record and review of the material presented for Dr. Andy Liu's 2003 course MATH 164: Higher Algebra, at the University of Alberta. This was the second run of the course; the first having been the year before. Hall and Knight's chapters concerning the theory of equations were typeset and used with minor changes. After each chapter in Part 1 an analysis of highlights, problems, and corrections is included. These insights arise through working with actual students, the experience with whom was invaluable in producing Part 2. The commentaries are not the result of a formal study, but rather a collection of observations generated through experience.

Part 2 of the thesis is an improved version of the text. The improvements follow the lead of the comments in Part 1 and add additional material. We attempt to introduce ideas only when they are needed and highlight only the important results. The exercises have been placed more appropriately and the flow of the chapters more greatly emphasized.

---

[1] "Mathematics is absolutely integrated with Western civilization, which conquered and dominated the entire world. The acceptance, forced or voluntary, of Western knowledge, behavior and values, can not be associated with ideas like "the winner is the best, the losers are to be discarded". More than any other form of knowledge, mathematics is identified with winners. This is true in history, in the professions, in everyday life, in families, in schools. The only possibility of building up a planetary civilization depends on restoring the dignity of the losers and, together, winners and losers, moving into the new. This requires respect for each other. Otherwise, the efforts will be from the losers to become winners, and from the winners to protect themselves from the losers, thus generating defensive confrontation." – Ubiratan D'Ambrosio (in translation) From a talk entitled ETHNOMATHEMATICS: AN OVERVIEW delivered to the II Congresso Internacional de Etnomatemtica, 5-7 August 2002.

Subject to further improvements, Part 2 of the thesis will comprise a new release of Higher Algebra.

# Part 1

# CHAPTER I — COMPLEX NUMBERS

Although from the rule of signs it is evident that a negative quantity cannot have a real square root, imaginary quantities represented by symbols of the form $\sqrt{-a}$ and $\sqrt{-1}$ are of frequent occurrence in mathematical investigations, and their use leads to valuable results. We therefore proceed to explain in what sense such roots are to be regarded.

When the quantity under the radical sign is negative, we can no longer consider the symbol $\sqrt{\phantom{a}}$ as indicating a possible arithmetical operation; but just as $\sqrt{a}$ may be defined as a symbol which obeys the relation $\sqrt{a}\sqrt{a} = a$, so we shall define $\sqrt{-a}$ to be such that $\sqrt{-a}\sqrt{-a} = -a$, and we shall accept the meaning to which this assumpion leads us.

It will be found that this definition will enable us to bring imaginary quantities under the dominion of ordinary algebraical rules, and that through their use results may be obtained which can be relied on with as much certainty as others which depend solely on the use of real quantities.

By definition, $\sqrt{-1}\sqrt{-1} = -1$. Therefore, $\sqrt{a}\sqrt{-1}\sqrt{a}\sqrt{-1} = a(-1)$. That is, $(\sqrt{a}\sqrt{-1})^2 = -a$. Thus the product $\sqrt{a}\sqrt{-1}$ may be regarded as equivalent to the imaginary quantity $\sqrt{-a}$.

It will be convenient to denote $\sqrt{-1}$ by the symbol $i$. The imaginary character of an expression will be denoted by its presence. For instance, $\sqrt{-4} = 2i$ and $\sqrt{-7a^2} = a\sqrt{7}i$.

We shall always consider that, in the absence of any statement to the contrary, of the signs which may be prefixed before a radical the positive sign is to be taken. But in the use of imaginary quantities, there is one point of importance which deserves notice.

Since $(-a)(-b) = ab$, by taking the square root, we have $\sqrt{-a}\sqrt{-b} = \pm\sqrt{ab}$. Thus in forming the product of $\sqrt{-a}$ and $\sqrt{-b}$ it would appear that either of the signs $+$ or $-$ might be placed before $\sqrt{ab}$. This is not the case however, for $\sqrt{-a}\sqrt{-b} = \sqrt{a}i\sqrt{b}i = \sqrt{ab}i^2 = -\sqrt{ab}$.

A number of the form $a + bi$ is called a **complex number**. Here $a$ and $b$ are real numbers, but not necessarily rational. We call $a$ the real part and $b$ the imaginary part of the complex number $a + bi$.

In dealing with complex numbers, the usual rules or arithmetic apply.

**Addition/Subtraction Rule.**

$$(a + bi) \pm (c + di) = a \pm c + (b \pm d)i.$$

**Multiplication Rule.**

$$(a + bi) \cdot (c + di) = ac - bd + (ad + bc)i.$$

**Theorem 1.**
If $a + bi = 0$, then $a = 0$ and $b = 0$.

**Proof:**

Suppose $a + bi = 0$. Then $bi = -a$. Hence $-b^2 = a^2$ so that $a^2 + b^2 = 0$. Now $a^2$ and $b^2$ are both non-negative; therefore their sum cannot be zero unless each of them is zero. That is, $a = 0$ and $b = 0$.

**Theorem 2.**
If $a + bi = c + di$, then $a = c$ and $b = d$.

**Proof:**

By transposition, $a - c + (b - d)i = 0$. By Theorem 1, $a - c = 0$ and $b - d = 0$; that is, $a = c$ and $b = d$.

*Thus in order that two complex numbers be equal it is necessary and sufficient that the real parts be equal and the imaginary parts be equal.*

**Definition.**
When two complex numbers differ only in the sign of the imaginary part, they are said to be *conjugate.*

For instance, $2 - 3i$ is conjugate to $2 + 3i$. In general, $a - bi$ is conjugate to $a + bi$.

**Theorem 3.**
The sum and product of two conjugate complex numbers are both real.

**Proof:**

We have $(a+bi)+(a-bi) = 2a$ and $(a+bi)(a-bi) = a^2 - (-b^2) = a^2 + b^2$.

**Definition.**
The positive value of the square root of $a^2 + b^2$ is called the *modulus* of each of the conjugate complex numbers $a + bi$ and $a - bi$.

**Theorem 4.**
The modulus of the product of two complex numbers is equal to the product of their moduli; that is,: modulus$(x \cdot y)$ =modulus$(x)\cdot$modulus$(y)$.

**Proof:**

Let the two complex numbers be denoted by $a + bi$ and $c + di$. Then their product is $ac - bd + (ad + bc)i$, which is a complex number whose modulus is

$$\begin{aligned}\sqrt{(ac-bd)^2 + (ad+bc)^2} &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}.\end{aligned}$$

This proves the proposition.

If the denominator of a fraction is of the form $a+bi$, it may be rationalized by multiplying the numerator and the denominator by the conjugate expression $a - bi$.

For instance,

$$\begin{aligned}\frac{c+di}{a+bi} &= \frac{(c+di)(a-bi)}{(a+bi)(a-bi)} \\ &= \frac{ac+bd+(ad-bc)i}{a^2+b^2} \\ &= \left(\frac{ac+bd}{a^2+b^2}\right) + \left(\frac{ad-bc}{a^2+b^2}\right)i.\end{aligned}$$

Thus we see that the sum, difference, product and quotient of two complex numbers is in each case another complex number.

**Example 1.**
Reduce $\frac{(2+3i)^2}{2+i}$ to the form $A + Bi$.

**Solution:**

The given expression is equal to

$$\begin{aligned}\frac{4-9+12i}{2+i} &= \frac{(-5+12i)(2-i)}{(2+i)(2-i)} \\ &= \frac{-10+12+29i}{4+1} \\ &= \frac{2}{5} + \frac{29}{5}i,\end{aligned}$$

which is of the required form.

## Square Roots

We wish to find the square root of $a+bi$. Assume that it is equal to $x+yi$. where $x$ and $y$ are real quantities. By squaring, $a + bi = x^2 - y^2 + 2xyi$. Therefore, by equating real and imaginary parts, $x^2 - y^2 = a$ and $2xy = b$. Now, $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = a^2 + b^2$. So $x^2 + y^2 = \sqrt{a^2+b^2}$. From this and $x^2 - y^2 = a$, we obtain $x^2 = \frac{\sqrt{a^2+b^2}+a}{2}$ and $y^2 = \frac{\sqrt{a^2+b^2}-a}{2}$. Finally, $x = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}$ and $y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$.

**Remark:**
Since $x$ and $y$ are real numbers, $x^2 + y^2$ is positive, and therefore the positive root is chosen in $x^2 + y^2 = \sqrt{a^2+b^2}$. Also from $2xy = b$, we see that the product $xy$ must have the same sign as $b$; hence $x$ and $y$ must have like signs if $b$ is positive, and unlike signs if $b$ is negative.

**Example 2.**
Find the square root of $-7 - 24i$.

**Solution:**

Assume $\sqrt{-7-24i} = x + yi$; then $-7 - 24i = x^2 - y^2 + 2xyi$. Therefore $x^2 - y^2 = -7$ and $2xy = -24$. Now $(x^2+y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = 49 + 576 = 625$. Hence $x^2 + y^2 = 25$. From these equations one can see $x^2 = 9$ and $y^2 = 16$ and therefore $x = \pm 3$ and $y = \pm 4$. However, since the product $xy$ is negative we must take $x = 3$ and $y = -4$, or $x = -3$ and $y = 4$. Finally, $\sqrt{-7-24i} = \pm(3-4i)$.

**Example 3.**
Find the value of $\sqrt[4]{-64a^4}$.

**Solution:**

Note that $\sqrt[4]{-64a^4} = \sqrt{\pm 8a^2 i} = 2a\sqrt{2}\sqrt{\pm i}$. It remains to find the value of $\sqrt{\pm i}$. Assume $\sqrt{i} = x + yi$. Then $i = x^2 - y^2 + 2xyi$. Therefore $x^2 - y^2 = 0$ and $2xy = 1$, whence $x = \frac{1}{\sqrt{2}}$ and $y = \frac{1}{\sqrt{2}}$, or $x = -\frac{1}{\sqrt{2}}$ and $y = -\frac{1}{\sqrt{2}}$. Hence $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1+i)$. Similarly, $\sqrt{-i} = \pm\frac{1}{\sqrt{2}}(1-i)$ and so $\sqrt{\pm i} = \pm\frac{1}{\sqrt{2}}(1 \pm i)$. Finally $\sqrt[4]{-64a^4} = \pm 2a(1 \pm i)$.

It is useful to notice the successive powers of $i$; thus $i^1 = i$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. Since each power is obtained by multiplying the one before it by $i$, we see that the results must now recur in a cycle with these four steps.

## Cube Roots

In this subsection, we shall investigate the properties of certain complex numbers which are of very frequent occurrence.

Suppose $x = \sqrt[3]{1}$; then $x^3 = 1$, or $x^3 - 1 = 0$; that is, $(x-1)(x^2+x+1) = 0$. Therefore, either $x - 1 = 0$, or $x^2 + x + 1 = 0$; whence $x = 1$ or $x = \frac{-1\pm\sqrt{3}i}{2}$.

It may be shown by actual computation that each of these values when cubed is equal to unity. Thus unity has three cube roots, 1, $x = \frac{-1+\sqrt{3}i}{2}$ and $x = \frac{-1-\sqrt{3}i}{2}$, two of which are complex numbers.

Let us denote these by $\alpha$ and $\beta$; then since they are the roots of the equation $x^2 + x + 1 = 0$, their product is equal to unity. We have then $\alpha\beta = 1$. Since $\alpha^3 = 1$, $\beta = \alpha^3\beta = \alpha^2$. Similarly we may show that $\alpha = \beta^2$.

Since each of the imaginary roots is the square of the other, it is usual to denote the three cube roots of unity by 1, $\omega$ and $\omega^2$.

Also $\omega$ satisfies the equation $x^2 + x + 1 = 0$; so $\omega^2 + \omega + 1 = 0$. In other words, *the sum of the three cube roots of unity is zero.* Again, $\omega \cdot \omega^2 = \omega^3 = 1$. Therefore,

(1) the product of the two imaginary roots is unity;

(2) every integral power of $\omega^3$ is unity.

It is useful to notice that the successive positive integral powers of $\omega$ are 1, $\omega$ and $\omega^2$; for, if $n$ be a multiple of 3, it must be of the form $3m$, and $\omega^n = \omega^{3m} = 1$. Conversely, if $n$ be not a multiple of 3, it must be of the form $3m+1$ or $3m+2$. If $n = 3m+1$, $\omega^n = \omega^{3m+1} = \omega^{3m}\cdot\omega = \omega$. If $n = 3m+2$, $\omega^n = \omega^{3m+2} = \omega^{3m}\cdot\omega^2 = \omega^2$.

We now see that every real number has three cube roots, two of which are complex. For the cube roots of $a^3$ are those of $a^3 \cdot 1$, and therefore are $a$, $a\omega$ and $a\omega^2$. Similarly the cube roots of 9 are $\sqrt[3]{9}$, $\omega\sqrt[3]{9}$ and $\omega^2\sqrt[3]{9}$, where $\sqrt[3]{9}$ is the cube root found by the ordinary arithmetical rule. In future, unless otherwise stated, the symbol $\sqrt[3]{a}$ will always be taken to denote the arithmetical cube root of $a$.

**Example 4.**
Resolve $x^3 + y^3$ into three factors of the first degree.

**Solution:**

Since $\omega + \omega^2 = -1$ and $\omega^3 = 1$, we have

$$\begin{aligned} x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\ &= (x+y)(x+\omega y)(x+\omega^2 y). \end{aligned}$$

**Example 5.**
Show that $(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) = a^2 + b^2 + c^2 - bc - ca - ab$.

**Solution:**

In the product of $a + \omega b + \omega^2 c$ and $a + \omega^2 b + \omega$ the coefficients of $b^2$ and $c^2$ are $\omega^3 = 1$; the coefficient of $bc$ is $\omega^2 + \omega^4 = \omega^2 + \omega = -1$; and the coefficients of $ca$ and $ab$ are $\omega^2 + \omega = -1$.

**Example 6.**
Show that $(1 + \omega - \omega^2)^3 - (1 - \omega + \omega^2)^3 = 0$.

**Solution:**

Since $1 + \omega + \omega^2 = 0$ we have

$$\begin{aligned} (1+\omega-\omega^2)^3 - (1-\omega+\omega^2)^3 &= (-2\omega^2)^3 - (-2\omega)^3 \\ &= -8\omega^6 + 8\omega^3 \\ &= -8 + 8 \\ &= 0. \end{aligned}$$

## EXERCISES I

Compute:

1. $(2\sqrt{-3} + 3\sqrt{-2})(4\sqrt{-3} - 5\sqrt{-2})$.

2. $(3\sqrt{-7} - 5\sqrt{-2})(3\sqrt{-7} - 5\sqrt{-2})$.

3. $(e^i + e^{-i})(e^i - e^{-i})$.

4. $(x - \frac{1+\sqrt{-3}}{2})(x - \frac{1-\sqrt{-3}}{2})$.

Express with rational denominator:

5. $\frac{1}{3-\sqrt{-2}}$.

6. $\frac{3\sqrt{-2}+2\sqrt{-5}}{3\sqrt{-2}-2\sqrt{-5}}$.

7. $\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$.

8. $\frac{a+xi}{a-xi} - \frac{a-xi}{a+xi}$.

9. $\frac{(x+i)^2}{x-i} - \frac{(x-i)^2}{x+i}$.

10. $\frac{(a+i)^3-(a-i)^3}{(a+i)^2-(a-i)^2}$.

11. Find the value of $(-i)^{4n+3}$, when $n$ is a positive integer.

12. Find the square of $\sqrt{9+40i} + \sqrt{9-40i}$.

Find the square root of:

13. $-5 + 12i$.

14. $-11 - 60i$.

15. $-47 + 8\sqrt{3}i$.

16. $-8i$.

17. $a^2 - 1 + 2ai$.

18. $4ab - 2(a^2 - b^2)i$.

Express in the form $A + Bi$:

19. $\frac{3+5i}{2-3i}$.

20. $\frac{\sqrt{3}-\sqrt{2}i}{2\sqrt{3}-\sqrt{2}i}$.

21. $\frac{1+i}{1-i}$.

22. $\frac{(1+i)^2}{3-i}$.

23. $\frac{(a+bi)^2}{a-bi} - \frac{(a-bi)^2}{a+bi}$.

If 1, $\omega$ and $\omega^2$ are the three cube roots of unity, prove that:

24. $(1 + \omega^2)^4 = \omega$.

25. $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$.

26. $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^5) = 9$.

27. $(2 + 5\omega + 2\omega^2)^6 = (2 + 2\omega + 5\omega^2)^6 = 729$.

28. $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8) \cdots (1 - \omega^n + \omega^{2n}) = 2^{2n}$.

29 Prove that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$.

30. If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$, show that

    (a) $xyz = a^3 + b^3$;

    (b) $x^2 + y^2 + z^2 = 6ab$;

    (c) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$.

31. If $ax + cy + bz = X$, $cx + by + az = Y$, $bx + ay + cz = Z$, show that

$$(a^2 + b^2 + c^2 - bc - ca - ab)(x^2 + y^2 + z^2 - yz - zx - xy)$$

$$= X^2 + Y^2 + Z^2 - YZ - XZ - XY.$$

### Solutions to Even-numbered Exercises I

2. We have $(3\sqrt{7}i - 5\sqrt{2}i)(3\sqrt{7}i + 5\sqrt{2}i) = i^2((3\sqrt{7})^2 - (5\sqrt{2})^2) = -(63 - 50) = -13$.

4. We have $(x - \frac{1+\sqrt{3}i}{2})(x - \frac{1-\sqrt{3}i}{2}) = x^2 - (\frac{1+\sqrt{3}i}{2} + \frac{1-\sqrt{3}i}{2}) + \frac{1^2-(\sqrt{3}i)^2}{2^2} = x^2 - x + 1$.

6. We have $\frac{3\sqrt{2}i+2\sqrt{5}i}{3\sqrt{2}i-2\sqrt{5}i} = \frac{3\sqrt{2}+2\sqrt{5}}{3\sqrt{2}-2\sqrt{5}} \times \frac{3\sqrt{2}+2\sqrt{5}}{3\sqrt{2}+2\sqrt{5}} = \frac{38+12\sqrt{10}}{-2} = -19 - 6\sqrt{10}$.

8. We have $\frac{a+xi}{a-xi} - \frac{a-xi}{a+xi} = \frac{(a+xi)^2-(a-xi)^2}{a^2-(xi)^2} = \frac{4axi}{a^2+x^2}$.

10. We have $\frac{(a+i)^3-(a-i)^3}{(a+i)^2-(a-i)^2} = \frac{6a^2i-2i}{4ai} = \frac{3a^2-1}{2a}$.

12. We have $(\sqrt{9 + 40i} + \sqrt{9 - 40i})^2 = 9 + 40i + 9 - 40i + 2\sqrt{9^2 - (40i)^2} = 18 + 2\sqrt{1681} = 100$.

14. Let $\sqrt{-11 - 60i} = x + yi$. Then $-11 - 60i = x^2 - y^2 + 2xyi$. Hence $x^2 - y^2 = -11$ and $xy = -30$. Now $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4(xy)^2 = 3721$, so that $x^2 + y^2 = 61$. Combined with $x^2 - y^2 = -11$, we have $x^2 = 25$ and $y^2 = 36$. Since $xy = -30$, $x$ and $y$ are of opposite signs. Hence $\sqrt{-11 - 60i} = \pm(5 - 6i)$.

16. Let $\sqrt{-8i} = x + yi$. Then $-8i = x^2 - y^2 + 2xyi$. Hence $x^2 - y^2 = 0$ and $xy = -4$. From $x^2 - y^2 = 0$, either $y = x$ or $y = -x$. Since $xy = -4$, we must have $y = -x$. Hence $-x^2 = -4$ so that $x = \pm 2$ and $y = \mp 2$. It follows that $\sqrt{-8i} = \pm 2(1 - i)$.

18. Let $\sqrt{4ab-2(a^2-b^2)i}=x+yi$. Then $4ab-2(a^2-b^2)i=x^2-y^2+2xyi$. Hence $x^2-y^2=4ab$ and $xy=-2(a^2-b^2)$. Now $(x^2+y^2)^2=(x^2-y^2)^2+4(xy)^2=16a^2b^2+4a^4+4b^4-8a^2b^2$, so that $x^2+y^2=2(a^2+b^2)$. Combined with $x^2-y^2=4ab$, we have $x^2=a^2+b^2+2ab$ and $y^2=a^2+b^2-2ab$. Since $xy=-2(a^2-b^2)$, $x$ and $y$ are of opposite signs. Hence $\sqrt{4ab-2(a^2-b^2)i}=\pm(a+b-(a-b)i)$.

20. We have $\frac{\sqrt{3}-\sqrt{2}i}{2\sqrt{3}-\sqrt{2}i}\times\frac{2\sqrt{3}+\sqrt{2}i}{2\sqrt{3}+\sqrt{2}i}=\frac{6+\sqrt{6}i-2\sqrt{6}i+2}{12+2}=\frac{4}{7}-\frac{\sqrt{6}}{14}i$.

22. We have $\frac{(1+i)^2}{3-i}\times\frac{3+i}{3+i}=\frac{2i(3+i)}{10}=-\frac{1}{5}+\frac{3}{5}i$.

24. We have $(1+\omega^2)^4=(-\omega)^4=\omega$ using $1+\omega+\omega^2=0$ and $\omega^3=1$.

26. Using $\omega^3=1$ and $1+\omega+\omega^2=0$, we have

$$\begin{aligned}
& (1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) \\
= {} & (1-\omega)^2(1-\omega^2)^2=(1-2\omega+\omega^2)(1-2\omega^2+\omega) \\
= {} & (-3\omega)(-3\omega^2) \\
= {} & 9.
\end{aligned}$$

28. The product of the first two factors is $(1-\omega+\omega^2)(1-\omega^2+\omega^4)=(-2\omega)(-2\omega^2)=2^2$. The product of the next two factors is $(1-\omega^{+}\omega^2)(1-\omega^2+\omega)$ again. It follows that the product of the $2n$ factors is $2^{2n}$.

30. (a) We have $a^3+b^3=(a+b)(a^2-ab+b^2)$, so that it is only necessary to prove that $yz=a^2-ab+b^2$. Indeed, we have

$$yz=a^2\omega^3+ab\omega^2+ba\omega^4+b^2\omega^3=a^2+ab(\omega^2+\omega)+b^2=a^2-ab+b^2.$$

(b) We have $x^2=a^2+2ab+b^2$, $y^2=a^2\omega^2+2ab\omega^3+b^2\omega^4$ and $z^2=a^2\omega^4+2ab\omega^3+b^2\omega^2$. Hence $x^2+y^2+z^2=a^2(1+\omega^2+\omega)+ab(1+1+1)+b^2(1+\omega+\omega^2)=3ab$.

(c) We have $y^3=a^3\omega^3+3a^2b\omega^4+3ab^2\omega^5+b^3\omega^6$, $z^3=a^3\omega^6+3a^2b\omega^5+3ab^2\omega^4+b^3\omega^3$ and $x^3=a^3+3a^2b+3ab^2+b^3$. Hence

$$\begin{aligned}
x^3+y^3+z^3 &= a^3(1+1+1)+3a^2b(1+\omega+\omega^2) \\
&\quad +3ab^2(1+\omega^2+\omega)+b^3(1+1+1) \\
&= 3(a^3+b^3).
\end{aligned}$$

### Answers to Odd-numbered Exercises I

1. $6-2\sqrt{6}$.

3. $e^{2i}-e^{-2i}$.

5. $\frac{3}{11}+\frac{\sqrt{2}}{11}i$.

7. $-\frac{8}{29}$.

9. $\frac{2(3x^2-1)}{x^2+1}i$.

11. $i$.

13. $\pm(2+3i)$. 15. $\pm(1+4\sqrt{3}i)$. 17. $\pm(a+i)$.

19. $-\frac{9}{13}+\frac{19}{13}i$. 21. $i$. 23. $\frac{2b(3a^2-b^2)}{a^2+b^2}i$.

## Commentary I

Complex numbers was chosen for the opening chapter because we felt the topic would be appealing and new to most students while not presenting too much of a technical challenge. Many students had heard of imaginary numbers and were needless to say both skeptical and curious. This chapter sets the tone for the course by including reoccuring themes: introductory exposition, definitions and theorems which establish the algebraic framework, the demonstation of a special artifice to obtain a result (square roots here), and some examples which involve substantial algebraic skill.

The artifice used to solve for $x$ and $y$ given $x^2-y^2$ and $xy$ presented some difficulty. A special section on this maneuver has been included in Chapter 0 where it is placed in the broader theory of symmetric functions.

Although the roots of unity topic was popular with the stronger students we have moved this section into the chapter on cubics as it is not necessary until then. The new approach is to let the equations lead to the theory. In this way the complex numbers chapter is introduced by the equation $x^2+1=0$

Theorem 1 has been downgraded to a lemma and Theorem 4 has become an exercise because the result is not actually used. This pruning has the effect of highlighting the important theorems. Example 3 presented some problems and was probably unnecessarily complicated. The new example is the (trick?) question $\sqrt{i}$

In Hall and Knight's original text the symbol $\sqrt{-1}$ was retained for the entire chapter. We chose instead to introduce the $i$ notation immediately, but noticed that a common mistake was,

$$\sqrt{-35}=\sqrt{35i}.$$

It was not always clear whether this was sloppy notation or sloppy thinking because the next step often corrected the error. We stand by the decision to use the $i$ notation, but caution that this error should be pointed out.

As it stands in Part 1, this chapter is a nice introduction to complex numbers and some of the manipulations that concern them. In the new version this introduction is placed more firmly in the context of solving quadratic equations.

## CHAPTER II — THE THEORY OF QUADRATIC EQUATIONS

The general form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$. We assume for now that it always have two roots, and prove only the following partial result.

**Theorem 1.**
A quadratic equation cannot have three different roots.

**Proof:**

Assume to the contrary that it has three roots different roots $\alpha, \beta$ and $\gamma$. Then we have

$$\begin{aligned} a\alpha^2 + b\alpha + c &= 0, \qquad (1) \\ a\beta^2 + b\beta + c &= 0, \qquad (2) \\ a\gamma^2 + b\gamma + c &= 0. \qquad (3) \end{aligned}$$

From (1) and (2), by subtraction, $a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0$. Divide out by $\alpha - \beta$ which, by hypothesis, is not zero; then

$$a(\alpha + \beta) + b = 0. \tag{4}$$

Similarly from (2) and (3),

$$a(\beta + \gamma) + b = 0. \tag{5}$$

From (4) and (5), $a(\alpha - \gamma) = 0$; which is impossible, since by hypothesis, $a \neq 0$ and $\alpha \neq \gamma$. Hence there cannot be three different roots.

We now come to an important result.

**Theorem 2.**
If $\alpha$ and $\beta$ are the roots of $ax^2 + bx + c = 0$, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

**Proof:**

Clearly, $\alpha$ and $\beta$ are the roots of $(x - \alpha)(x - \beta) = 0$. This expands into $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. Comparing with $x^2 + \frac{b}{a}x + \frac{c}{a}$, we have the desired results.

In fact, any quadratic equation may be expressed in the form

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0.$$

So, given the roots, we can easily form an equation.

**Example 1.**
Form an equation whose roots are 3 and $-2$.

**Solution:**

The equation is $(x - 3)(x + 2) = 0$ or $x^2 - x + 6 = 0$.

When the given roots are irrational, it is easier to use the following method.

**Example 2.**
Form an equation whoose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

**Solution:**

We have sum of roots $=4$ and product of roots $=1$; so the equation is $x^2-4x+1=0$.

By a method analogous to that used in Example 1, we can form an equation with three or more given roots.

**Example 3.**
Form an equation whose roots are 2, $-3$ and $\frac{7}{5}$.

**Solution:**

The required equation must be satisfied by each of the following suppositions: $x-2=0$, $x+3=0$ and $x-\frac{7}{5}=0$. Therefore the equation must be $(x-2)(x+3)(x-\frac{7}{5})=0$, which may be rewritten as $(x-2)(x+3)(5x-7)=5x^3-2x^2-37x+42=0$.

**Example 4.**
Form an equation whose roots are 0, $\pm a$ and $\frac{c}{b}$.

**Solution:**

The equation has to be satisfied by $x=0$, $x=a$, $x=-a$ and $x=\frac{c}{b}$. Therefore it is $x(x-a)(x+a)(x-\frac{c}{b})=0$. This may be rewritten as $x(x^2-a^2)(bx-c)=bx^4-cx^3-a^2bx^2+a^2cx=0$.

Theorem 2 is generally sufficient to solve problems connected with the roots of quadratic equations. In such questions the roots should never be considered singly, but use should be made of the relations obtained by writing down the sum of the roots, and their product, in terms of the coefficients of the equation.

**Example 5.**
If $\alpha$ and $\beta$ are the roots of $x^2-px+q=0$, find the value of

(a) $\alpha^2+\beta^2$;

(b) $\alpha^3+\beta^3$.

**Solution:**

We have $\alpha+\beta=p$ and $\alpha\beta=q$.

(a) We have $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-2q$.

(b) We have $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)=p((\alpha+\beta)^2-3\alpha\beta)=p(p^2-3q)$.

**Example 6.**
If $\alpha$ and $\beta$ are the roots of the equation $\ell x^2 + mx + n = 0$, find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

**Solution:**

We have sum of roots $= \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta}$ and product of roots $= \frac{\alpha}{\beta}\frac{\beta}{\alpha} = 1$. By Theorem 2, the required equation is $x^2 - (\frac{\alpha^2+\beta^2}{\alpha\beta})x + 1 = 0$, which may be rewritten as $\alpha\beta x^2 - (\alpha^2+\beta^2)x + \alpha\beta = 0$. As in the last example, $\alpha^2 + \beta^2 = \frac{m^2-2n\ell}{\ell^2}$ and $\alpha\beta = \frac{n}{\ell}$. So the equation is $\frac{n}{\ell}x^2 - \frac{m^2-2n\ell}{\ell^2}x + \frac{n}{\ell} = 0$, which may be rewritten as $n\ell x^2 - (m^2 - 2n\ell)x + n\ell = 0$.

**Example 7.**
Let $x = \frac{3+5i}{2}$.

(a) Find the value of $2x^3 + 2x^2 - 7x + 72$.

(b) Show that it will be unaltered if $\frac{3-5i}{2}$ is substituted for $x$ instead.

**Solution:**

Form the quadratic equation whose roots are $x = \frac{3\pm5i}{2}$. Since sum of roots=3 and product of roots= $\frac{17}{2}$, the equation is $2x^2 - 6x + 17 = 0$.

(a) Perform the following long division.

$$\begin{array}{rrrrrrr}
 & & & & & x & +4\\ \cline{4-7}
2x^2 & -6x & +17 \;) & 2x^3 & +2x^2 & -7x & +72\\
 & & & 2x^3 & -6x^2 & +17x & \\ \cline{4-7}
 & & & & 8x^2 & -24x & +72\\
 & & & & 8x^2 & -24x & +68\\ \cline{5-7}
 & & & & & & 4
\end{array}$$

When $\frac{3+5i}{2}$ is substituted for $x$, $2x^2 - 6x + 17 = 0$. Hence

$$2x^3 + 2x^2 - 7x + 72 = (x+4)(2x^2 - 6x + 17) + 4 = 4.$$

(b) Note that $2x^2 - 6x + 17$ is a quadratic *expression* which vanishes for *either* of the values $\frac{3\pm5i}{2}$. Hence when $\frac{3-5i}{2}$ is substituted for $x$, the value of $2x^3 + 2x^2 - 7x + 72$ will be unaltered.

We now present two minor results about quadratic equations whose roots are related to each other in a special way.

**Theorem 3.**
If the roots of the equation $ax^2 + bx + c = 0$ are equal in magnitude and opposite in signs, then $b = 0$.

**Proof:**

The roots will be equal in magnitude and opposite in sign if their sum is zero. Hence the required condition is $-\frac{b}{a} = 0$, or equivalently $b = 0$.

**Theorem 4.**
If the roots of the equation $ax^2 + bx + c = 0$ are reciprocals of each other, then $c = a$.

**Proof:**

The roots will be reciprocals of each other when their product is 1. Hence the required condition is $\frac{c}{a} = 1$, or equivalently, $c = a$.

Theorem 3 is of frequent occurrence in Analytical Geometry, and Theorem 4 is a particular case of a more general condition applicable to equations of any degree.

**Example 8.**
Find the condition that the roots of $ax^2 + bx + c = 0$ may be

(a) both positive;

(b) opposite in sign, with the negative one numerically greater.

**Solution:**

Let the roots be $\alpha$ and $\beta$. Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(a) If the roots are both positive, $\alpha\beta$ is positive. Therefore $c$ and $a$ have like signs. Also, since $\alpha + \beta$ is positive, $\frac{b}{a}$ is negative. Therefore $b$ and $a$ have unlike signs. In summary, the required condition is that the signs of $a$ and $c$ should be the same, and opposite to the sign of $b$.

(b) If the roots are of opposite signs, $\alpha\beta$ is negative. Therefore $c$ and $a$ have unlike signs. Also since $\alpha + \beta$ has the sign of the numerically greater root, it is negative. Therefore $\frac{b}{a}$ is positive and $b$ and $a$ have like signs. In summary, the required condition is that the signs of $a$ and $b$ should be the same, and opposite to the sign of $c$.

## Quadratic Formula

Let $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$. Then

$$\alpha + \beta = -\frac{b}{a}; \tag{6}$$

$$\alpha\beta = \frac{c}{a}. \tag{7}$$

Hence $(\alpha-\beta)^2 = (\alpha+\beta)^2 - 4\alpha\beta = (-\frac{b}{a})^2 - 4(\frac{c}{a}) = \frac{b^2-4ac}{a^2}$ by (6) and (7). We may take

$$\alpha - \beta = \frac{\sqrt{b^2-4ac}}{a}. \tag{8}$$

From (6) and (8), we have $\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$. Hence the roots of $ax^2+bx+c=0$ are $x = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$. This is known as the **Quadratic Formula**.

We can now give a direct verification of Theorem 2. We have

$$\begin{aligned} \alpha+\beta &= \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} \\ &= -\frac{2b}{2a} \\ &= -\frac{b}{a} \end{aligned}$$

and

$$\begin{aligned} \alpha\beta &= \frac{-b-\sqrt{b^2-4ac}}{2a}\frac{-b+\sqrt{b^2-4ac}}{2a} \\ &= \frac{(-b)^2-(b^2-4ac)}{4a^2} \\ &= \frac{4ac}{4a^2} \\ &= \frac{c}{a}. \end{aligned}$$

By writing the equation in the form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, these results may also be expressed as follows. In a quadratic equation where the coefficient of the first term is 1,

(i) the sum of the roots is equal to the coefficient of $x$ with its sign changed;

(ii) the product of the roots is equal to the third term.

**Note**. In any equation the term which does not contain the unknown quantity is frequently called *the constant term.*

The expression $b^2-4ac$ under the radical is called the **discriminant** of the equation $ax^2+bx+c=0$, because the nature of its roots may be determined without solving the equation. We simply apply the following tests.

Let $\alpha$ and $\beta$ be the roots of $ax^2+bx+c=0$.

(i) If $b^2-4ac$ is positive, $\alpha$ and $\beta$ are real and unequal.

(ii) If $b^2 - 4ac = 0$, then $\alpha = \beta = \frac{b}{2a}$.

(iii) If $b^2 - 4ac$ is negative, $\alpha$ and $\beta$ are conjugate complex numbers.

(iv) If $b^2 - 4ac$ is a square, $\alpha$ and $\beta$ are rational.

**Example 9.**
Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of $x$.

**Solution:**

Here $a = 2$, $b = -6$ and $c = 7$, so that $b^2 - 4ac = (-6)^2 - 4 \cdot 2 \cdot 7 = -20$. Therefore the roots are not real.

**Example 10.**
If the equation $x^2 + 2(k+2)x + 9k = 0$ has equal roots, find $k$.

**Solution:**

Using the condition for equal roots, $b^2 - 4ac = 0$ implies $(k+2)^2 = 9k$. This may be rewritten as $k^2 - 5k + 4 = 0$ or $(k-4)(k-1) = 0$. Hence $k = 4$ or $k = 1$.

**Example 11.**
Show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.

**Solution:**

The roots will be rational provided $(-2p)^2 - 4(p^2 - q^2 + 2qr - r^2)$ is a square. But this expression reduces to $4(q^2 - 2qr + r^2)$, or $4(q-r)^2$. Hence the roots are rational.

## EXERCISES II

Form the equations whose roots are:

1. $-\frac{4}{5}, \frac{3}{7}$.
2. $\frac{m}{n}, -\frac{n}{m}$.
3. $\frac{p-q}{p+q}, -\frac{p+q}{p-q}$.
4. $7 \pm 2\sqrt{5}$.
5. $\pm 2\sqrt{3} - 5$.
6. $-p \pm 2\sqrt{2q}$.
7. $-3 \pm 5i$.
8. $-a \pm bi$.
9. $\pm(a-b)i$.
10. $-3, \frac{2}{3}, \frac{1}{2}$.
11. $\frac{a}{2}, 0, -\frac{2}{a}$.
12. $2 \pm \sqrt{3}, 4$.

13. Prove that the roots of the following equations are real:

(a) $x^2 - 2ax + a^2 - b^2 - c^2 = 0$;

(b) $(a - b + c)x^2 + 4(a - b)x + (a - b - c) = 0$.

14. If the equation $x^2 - 15 - m(2x - 8) = 0$ has equal roots, find the values of $m$.

15. For what values of $m$ will the equation $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ have equal roots?

16. For what value of $m$ will the equation $\frac{x^2 - bx}{ax - c} = \frac{m-1}{m+1}$ have roots equal in magnitude but opposite in sign?

17. Prove that the roots of the following equations are rational:

(a) $(a + c - b)x^2 + 2cx + (b + c - a) = 0$;

(b) $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$.

If $\alpha$ and $\beta$ are the roots of the equation $ax^2 + bx + c = 0$, find the values of

18. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

19. $\alpha^4\beta^7 + \alpha^7\beta^4$.

20. $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$.

Find the value of

21. $x^3 + x^2 - x + 22$ when $x - 1 + 2i$.

22. $x^3 - 3x^2 - 8x + 15$ when $x = 3 + i$.

23. $x^3 - ax^2 + 2a^2x + 4a^3$ when $\frac{x}{a} = 1 - \sqrt{-3}$.

24. If $\alpha$ and $\beta$ are the roots of $x^2 + px + q = 0$, form the equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

25. Prove that the roots of $(x - a)(x - b) = h^2$ are always real.

26. If $\alpha$ and $\beta$ are the roots of $ax^2 + bx + c = 0$, find the value of

(a) $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$; (b) $(a\alpha + b)^{-3} + (a\beta + b)^{-3}$.

27. Find the condition that one root of $ax^2 + bx + c = 0$ shall be $n$ times the other.

28. If $\alpha$, $\beta$ are the roots of $ax^2 + bx + c = 0$, form the equation whose roots are $\alpha^2 + \beta^2$ and $\alpha^{-2} + \beta^{-2}$.

29. Form the equation whose roots are the squares of the sum and of the difference of the roots of $2x^2 + 2(m + n)x + m^2 + n^2 = 0$.

30. Discuss the signs of the roots of the equation $px^2 + qx + r = 0$.

### Solutions to Even-numbered Exercises II

2. The desired equation is $0 = (x - \frac{m}{n})(x - (-\frac{n}{m})) = x^2 - (\frac{m}{n} + (-\frac{n}{m}))x + \frac{m}{n}(-\frac{n}{m}) = x^2 - \frac{m^2-n^2}{mn} - 1$, or $mnx^2 + (n^2 - m^2)x - 1$.

4. The desired equation is

$$\begin{aligned} 0 &= (x - (7 + 2\sqrt{5}))(x - (7 - 2\sqrt{5})) \\ &= x^2 - (7 + 2\sqrt{5} + 7 - 2\sqrt{5})x + (7 + 2\sqrt{5})(7 - 2\sqrt{5}) \\ &= x^2 - 14x + 29. \end{aligned}$$

6. The desired equation is

$$\begin{aligned} 0 &= (x - (-p + 2\sqrt{2q}))(x - (-p - 2\sqrt{2q})) \\ &= x^2 - (-p + 2\sqrt{2q} - p - 2\sqrt{2q})x + (-p + 2\sqrt{2q})(-p - 2\sqrt{2q}) \\ &= x^2 + 2px + p^2 - 8q. \end{aligned}$$

8. The desired equation is

$$\begin{aligned} 0 &= (x - (-a + bi))(x - (-a - bi)) \\ &= x^2 - (-a + bi - a - bi)x + (-a + bi)(-a - bi) \\ &= x^2 + 2ax + a^2 + b^2. \end{aligned}$$

10. The desired equation is

$$\begin{aligned} 0 &= (x-(-3))(x-\frac{2}{3})(x-\frac{1}{2}) \\ &= x^3-(-3+\frac{2}{3}+\frac{1}{2})x^2+(-3(\frac{2}{3})-3(\frac{1}{2})+(\frac{2}{3})(\frac{1}{2})x-(-3)(\frac{2}{3})(\frac{1}{2}) \\ &= x^3+\frac{11}{6}x^2-\frac{19}{6}x+1, \end{aligned}$$

or $6x^3+11x^2-19x+6=0$.

12. The desired equation is

$$\begin{aligned} 0 &= (x-(2+\sqrt{3}))(x-(2-\sqrt{3}))(x-4) \\ &= x^3-(2+\sqrt{3}+2-\sqrt{3}+4)x^2+((2+\sqrt{3})(2-\sqrt{3}) \\ &\quad +4(2+\sqrt{3})+4(2-\sqrt{3}))x-4(2+\sqrt{3})(2-\sqrt{3}) \\ &= x^3-8x^2-17x-4. \end{aligned}$$

14. The equation may be rewritten as $x^2-2mx+(8m-15)=0$. It has equal roots if and only if $(-2m)^2=4(8m-15)$. This reduces to $m^2-8m+15=0$. The desired values are $m=\frac{1}{2}(8\pm\sqrt{8^2-4\times 15})=5$ or 3.

16. The equation may be rewritten as $(m+1)x^2-(b(m+1)+a(m-1))x+c(m-1)=0$. If the roots are $\pm r$, then the equation has the form $0=(x-r)(x-(-r))=x^2-r^2$. Hence $b(m+1)+a(m-1)=0$ so that $m=\frac{a-b}{a+b}$.

18. We have $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$. Hence $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-\frac{b}{a})^2-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$, so that $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}=\frac{b^2-2ac}{a^2}\div(\frac{c}{a})^2=\frac{b^2-2ac}{c^2}$.

20. From Problem 18, $\alpha\beta=-\frac{c}{a}$ and $\alpha^2+\beta^2=\frac{b^2-2ac}{a^2}$. Since $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2(\alpha\beta)^2$, it is equal to $(\frac{b^2-2ac}{a^2})^2-2(\frac{c}{a})^2=\frac{b^4-4ab^2c+2a^2c^2}{a^4}$. It follows that

$$\begin{aligned} \left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2 &= \left(\frac{\alpha^2-\beta^2}{\alpha\beta}\right)^2 \\ &= \frac{\alpha^4+\beta^4-2(\alpha\beta)^2}{(\alpha\beta)^2} \\ &= \frac{b^4-4ab^2c+2a^2c^2}{a^4}\div\frac{c^2}{a^2}-2 \\ &= \frac{b^2(b^2-4ac)}{a^2c^2}. \end{aligned}$$

22. We have $(3+i)^3-3(3+i)^2-8(3+i)+15=27+27i-9-i-27-18i+3-24-8i+15=-15$.

24. We have $\alpha+\beta = -p$ and $\alpha\beta = q$. Then $\alpha^2+\beta^2 = (\alpha+\beta)^2-2\alpha\beta = p^2-2q$. Hence $(\alpha-\beta)^2 + (\alpha+\beta)^2 = 2(\alpha^2+\beta^2) = 2(p^2-2q)$ and

$$(\alpha-\beta)^2(\alpha+\beta)^2 = (\alpha^2-\beta^2)^2 = (\alpha^2+\beta^2)^2-4(\alpha\beta)^2 = (p^2-2q)^2-4q^2 = p^2(p^2-4q).$$

It follows that the desired equation is $x^2-2(p^2-2q)x+p^2(p^2-4q) = 0$.

26. We have $\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. As in Problem 18, $\alpha^2+\beta^2 = \frac{b^2-2ac}{a^2}$. On the other hand, $\alpha^3+\beta^3 = (\alpha+\beta)^3-3\alpha\beta(\alpha+\beta) = (-\frac{b}{a})^3-3(-\frac{b}{a})(\frac{c}{a}) = \frac{b(3ac-b^2)}{a^3}$ while $(a\alpha+b)(a\beta+b) = a^2\alpha\beta+ab(\alpha+\beta)+b^2 = a^2(\frac{c}{a})+ab(-\frac{b}{a})+b^2 = ac$.

   (a) We have $(a\alpha+b)^2+(a\beta+b)^2 = a^2(\alpha^2+\beta^2)+2ab(\alpha+\beta)+2b^2$ which simplifies to $a^2(\frac{b^2-2ac}{a^2})+2ab(-\frac{b}{a})+2b^2 = b^2-2ac$. Hence $\frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2} = \frac{(a\alpha+b)^2+(a\beta+b)^2}{(a\alpha+b)^2(a\beta+b)^2} = \frac{b^2-2ac}{a^2c^2}$.

   (b) We have $(a\alpha+b)^3+(a\beta+b)^3 = a^3(\alpha^3+\beta^3)+3a^2b(\alpha^2+\beta^2)+3ab^2(\alpha+\beta)+b^3$ which simplifies to $a^3(\frac{b(3ac-b^2)}{a^3})+3a^2b(\frac{b^2-2ac}{a^2})+3ab^2(-\frac{b}{a})+2b^3 = b(b^2-3ac)$. Hence $\frac{1}{(a\alpha+b)^3}+\frac{1}{(a\beta+b)^3} = \frac{(a\alpha+b)^3+(a\beta+b)^3}{((a\alpha+b)(a\beta+b))^3} = \frac{b(b^2-3ac)}{a^3c^3}$.

28. From Problem 18, we have $\alpha\beta = \frac{c}{a}$ and $\alpha^2+\beta^2 = \frac{b^2-2ac}{a^2}$. The desired equation is

$$\begin{aligned} 0 &= (x-(\alpha^2+\beta^2))\left(x-\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right) \\ &= x^2-\left(\alpha^2+\beta^2+\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)x+(\alpha^2+\beta^2)\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right) \\ &= x^2-(\alpha^2+\beta+2)\frac{(\alpha\beta)^2+1}{(\alpha\beta)^2}x+\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)^2 \\ &= x^2-\frac{(b^2-2ac)(a^2+c^2}{a^2c^2}x+\frac{(b^2-2ac)^2}{a^2c^2}, \end{aligned}$$

or $a^2c^2x^2-(b^2-2ac)(a^2+c^2)x+(b^2-2ac)^2 = 0$.

30. Let the roots be $\alpha$ and $\beta$. We assume that they are real, and that $\alpha \geq \beta$. Their signs depend on those of their sum $-\frac{q}{p}$ and their product $\frac{r}{p}$. The following chart summarizes all scenarios.

| Signs | $pr < 0$ | $r = 0$ | $pr > 0$ |
|---|---|---|---|
| $pq < 0$ | $\alpha > 0 > \beta > -\alpha$ | $\alpha > \beta = 0$ | $\alpha > \beta > 0$ |
| $q = 0$ | $\alpha > 0 > \beta = -\alpha$ | $\alpha = \beta = 0$ | Imaginary roots |
| $pq > 0$ | $\alpha > 0 > -\alpha > \beta$ | $0 = \alpha > \beta$ | $0 > \alpha \geq \beta$ |

## Answers to Odd-numbered Exercises

1. $35x^2 + 13x - 12 = 0$. 3. $(p^2 - q^2)x^2 + 4pqx - (p^2 - q^2) = 0$.

5. $x^2 + 10x + 13 = 0$. 7. $x^2 + 6x + 34 = 0$.

9. $x^2 + (a - b)^2 = 0$. 11. $2ax^3 + (4 - a^2)x^2 - 2ax = 0$.

15. $2, -\frac{10}{9}$. 19. $\frac{bc^4(3ac-b^2)}{a^7}$. 21. 7. 23. 0.

27. $nb^2 = (1 + n)^2ac$. 29. $x^2 - 4mnx - (m^2 - n^2)^2 = 0$.

## Commentary II

Theorem 1, in addition to being only a partial result, fails to deal with the case of repeated roots. Moreover, results of this nature do not appeal to most students. In Part 2 we chose instead to emphasize Gauss' Fundamental Theorem of Algebra and point out that it uses more advanced methods. Theorem 2 was one of the highlights of the course. Its lack-luster name left something to be desired so we now call it Vieta's Theorem in the quadratic case as well as for the extension to general polynomials. This theorem has a simple proof that follows from the Fundamental Theorem nicely. It also ties together the well known quadratic factoring method with the theory of equations and roots and is even used to reprove the quadratic formula. When asked what their favorite theorem was at the end of the course many students responded enthusiastically, if not somewhat ambiguously, "Theorem 2!".

Examples 1 and 2 display two ways of doing the same question. Judgement, an important component in mathematical maturity, is called for in the choice of method. Example 3 illustrates how the idea can be extended beyond the quadratic case using the first method, but the example doesn't really belong in the chapter on quadratic equations.

Examples 5, 6, and 7 seem esoteric and unmotivated. Example 6 is a nice application of Theorem 2 and there should be no doubt that this is a difficult example. The trouble with Example 7 is that it can be solved with direct substitution. This is the method most often chosen by the students in the exercises despite this example's cleverness.

While most of the students must have seen the quadratic formula before, for many this was the first time they had seen a proof. This non-standard proof was certainly new to everyone. The proof uses the same mathod of sums and products as the method for extracting square roots of imaginary quatities; a method that is a theme in the course. In Part 2 the standard completing the square proof is also given. Hall and Knight present neither as they assume the material from their earlier *Elementary Algebra*.

# CHAPTER III — VALUES OF RATIONAL EXPRESSIONS

The following example illustrates a useful application of the results proved in the last chapter.

**Example 1.**
If $x$ is a real quantity, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6.

**Solution:**

Let $y = \frac{x^2+2x-11}{2(x-3)}$. This may be rewritten as $x^2 + 2x(1-y) + (6y-11) = 0$. This is a quadratic equation, and in order that $x$ may have real values the discriminant $b^2 - 4ac = 4(1-y)^2 - 4(6y-11)$ must be non-negative; or dividing by 4 and simplifying, $y^2 - 8y + 12 = (y-6)(y+2)$ must be non-negative. Hence the factors of this product must be both non-positive, or both non-negative. In the former case, $y \leq -2$, and in the latter case, $y \geq 6$. Therefore $y$ cannot lie between $-2$ and 6, but may have any other value.

In this example it will be noticed that the quadratic *expression* $y^2 - 8y + 12$ is non-negative so long as $y$ does not lie between the roots of the corresponding quadratic *equation* $y^2 - 8y + 12 = 0$. This is a particular case of the following general proposition.

**Theorem.**
For all real values of $x$ the expression $ax^2 + bx + c$ has the same sign as $a$, except when the roots of the equation $ax^2 + bx + c = 0$ are real and $x$ is equal to either or lies between them.

**Proof:**

CASE I.
Suppose that the roots of the equation $ax^2 + bx + c = 0$ are real and unequal. Denote them by $\alpha$ and $\beta$, and let $\alpha$ be the greater. Then

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a(x^2 - (\alpha+\beta)x + \alpha\beta) \\ &= a(x-\alpha)(x-\beta). \end{aligned}$$

Now if $x$ is greater than $\alpha$, the factors $x - \alpha$ and $x - \beta$ are both positive; and if $x$ is less than $\beta$, the factors $x - \alpha$ and $x - \beta$ are both negative. Therefore in each case the expression $(x-\alpha)(x-\beta)$ is positive, and $a^2 + bx + c$ has the same sign as $a$. But if $x$ has a value lying between $\alpha$ and $\beta$, the expression $(x-\alpha)(x-\beta)$ is negative, and the sign of $ax^2 + bx + c$ is opposite to that of $a$. Of course, if $x = \alpha$ or $x = \beta$, then $ax^2 + bx + c = 0$.

CASE II.
If the roots are real and both equal to $\alpha$, then $ax^2 + bx + c = a(x-\alpha)^2$, and

$(x-\alpha)^2$ is positive for all real values of $x \neq \alpha$. Hence $ax^2+bx+c$ has the same sign as $a$ unless $x=\alpha$, in which case its value is 0.

CASE III.

Suppose that the equation $ax^2+bx+c=0$ has complex roots. Then

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right). \end{aligned}$$

But $b^2-4ac$ is negative since the roots are complex. Hence $\frac{4ac-b^2}{4a^2}$ is positive, and the expression $(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a^2}$ is positive for all real values of $x$. Therefore $ax^2+bx+c$ has the same sign as $a$.

From the preceding result, it follows that the expression $ax^2+bx+c$ will always have the same sign whatever real value $x$ may have, provided that $b^2-4ac$ is negative. If this condition is satisfied, the expression is positive or negative according as $a$ is positive or negative.

Conversely, in order that the expression $ax^2+bx+c$ may be always positive, $b^2-4ac$ must be negative and $a$ must be positive; and in order that $ax^2+bx+c$ may be always negative, $b^2-4ac$ must be negative and $a$ must be negative.

**Example 2.**

Find the limits between which $a$ must lie in order that $\frac{ax^2-7x+5}{5x^2-7x+a}$ may be capable of all values, $x$ being any real quantity.

**Solution:**

Put $y=\frac{ax^2-7x+5}{5x^2-7x+a}$. Then $(a-5y)x^2-7x(1-y)+(5-ay)=0$. In order that the values of $x$ found from this quadratic may be real, the expression

$$49(1-y)^2-4(a-5y)(5-ay)=(49-20a)y^2+2(2a^2+1)y+(49-20a)$$

must be non-negative. Hence $(2a^2+1)^2-(49-20a)^2$ must be negative or zero, and $(49-20a)$ must be positive. Now

$$\begin{aligned} (2a^2+1)^2-(49-20a)^2 &= ((2a^2+1)-(49-20a))((2a^2+1)+(49-20a)) \\ &= (2a^2+20a-48)(2a^2-20a+50) \\ &= 4(a^2+10a-24)(a^2-10a+25) \\ &= 4(a+12)(a-2)(a-5)^2. \end{aligned}$$

The last expression is negative as long as $-12<a<2$, and for such values $49-20a$ is positive; the expression is zero when $a=-12$, 2 or 5, but $49-20a$ is negative when $a=5$. Hence $-12 \leq a \leq 2$.

# EXERCISES III

1. Determine the limits between which $n$ must lie in order that the equation

$$2ax(ax + nc) + (n^2 - 2)c^2 = 0$$

may have real roots.

2. If $x$ is real, prove that $-\frac{1}{11} \leq \frac{x}{x^2-5x+9} \leq 1$.

3. Show that $\frac{1}{3} \leq \frac{x^2-x+1}{x^2+x+1} \leq 3$ for all real values of $x$.

4. If $x$ is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

5. Find the equation whose roots are $\frac{\sqrt{a}}{\sqrt{a}\pm\sqrt{a-b}}$.

6. If $\alpha$ and $\beta$ are roots of the equation $x^2 - px + q = 0$, find the value of

   (a) $\alpha^2(\alpha^2\beta^{-1} - \beta) + \beta^2(\beta^2\alpha^{-1} - \alpha)$;
   (b) $(\alpha - p)^{-4} + (\beta - p)^{-4}$.

7. If the roots of $lx^2 + nx + n = 0$ are in the ratio of $p : q$, prove that $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} = \sqrt{\frac{n}{l}}$.

8. Show that if $x$ is real, the expression $\frac{(x+m)^2-4mn}{2(x-n)}$ admits of all values except such as lie between $2n$ and $2m$.

9. If the roots of the equation $ax^2 + 2bx + c = 0$ are $\alpha$ and $\beta$, and those of the equation $Ax^2 + 2Bx + C = 0$ are $\alpha + \delta$ and $\beta + \delta$, prove that $\frac{b^2-ac}{a^2} = \frac{B^2-AC}{A^2}$.

10. Show that the expression $\frac{px^2+3x-4}{p+3x-4x^3}$ will be capable of all values when $x$ is real, provided that $1 \leq p \leq 7$.

11. Find the greatest value of $\frac{x+2}{2x^2+3x+6}$ for real values of $x$.

12. Show that if $x$ is real, the expression $\frac{x^2-bc}{2x-b-c}$ has no real values between $b$ and $c$.

13. Prove that the roots of $ax^2+2bx+c = 0$ are real and unequal if and only if the roots of $(a + c)(ax^2 + 2bx + c) = 2(ac - b^2)(x^2 + 1)$ are complex.

14. Show that the expression $\frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will be capable of all values when $x$ is real, if $a^2 - b^2$ and $c^2 - d^2$ have the same sign.

## Solutions to Even-numbered Exercises III

2. Let $y = \frac{x}{x^2-5x+9}$. Then $yx^2 - (5y+1)x + 9y = 0$. In order for $x$ to be real, we must have $(5y+1)^2 - 4y(9y) \geq 0$. This is equivalent to $1 + 10y - 11y^2 \geq 0$ or $(1-y)(1+11y) \geq 0$. If $y > 1$, the first factor is negative but the second is positive. If $y < -\frac{1}{11}$, the first factor is positive but the second is negative. Hence we must have $-\frac{1}{11} \leq y \leq 1$.

4. Let $y = \frac{x^2+34x-71}{x^2+2x-7}$. Then $(y-1)x^2 + 2(y-17)x - (7y-71) = 0$ In order for $x$ to be real, we must have $4(y-17)^2 - 4(y-1)(7y-71) \geq 0$. This is equivalent to $8y^2 - 112y + 360 \geq 0$ or $8(y-5)(y-9) \geq 0$. If $5 < y < 9$, then $y - 5 > 0$ but $y - 9 < 0$. Hence $y$ has no value between 5 and 9.

6. We have $\alpha + \beta = p$ and $\alpha\beta = q$. Hence

$$\begin{aligned}
\alpha^2 + \beta^2 &= (\alpha+\beta)^2 - 2\alpha\beta = p^2 - q, \\
\alpha^3 + \beta^3 &= (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = p^3 - 3pq, \\
\alpha^4 + \beta^4 &= (\alpha+\beta)^4 - 4\alpha\beta(\alpha^2+\beta^2) - 6(\alpha\beta)^2 \\
&= p^4 - 4q(p^2-q) - 6q^2 = p^4 - 4p^2q + 2q^2
\end{aligned}$$

and

$$\begin{aligned}
\alpha^5 + \beta^5 &= (\alpha+\beta)^5 - 5\alpha\beta(\alpha^3+\beta^3) - 10(\alpha\beta)^2(\alpha+\beta) \\
&= p^5 - 5q(p^3 - 3pq) - 10pq^2 = p^5 - 5p^2q + 5pq^2.
\end{aligned}$$

(a) We have $\alpha^2(\frac{\alpha^2}{\beta} - \beta) + \beta^2(\frac{\beta^2}{\alpha} - \alpha) = \frac{\alpha^5+\beta^5-(\alpha\beta)^2(\alpha+\beta)}{\alpha\beta} = \frac{p^5-5p^2q+5pq^2-pq^2}{q} = \frac{p(p^2-q)(p^2-4q)}{q}$.

(b) We have

$$\begin{aligned}
&\frac{1}{(\alpha-p)^4} + \frac{1}{(\beta-p)^4} \\
=\ &\frac{\alpha^4+\beta^4-4p(\alpha^3+\beta^3)-6p^2(\alpha^2+\beta^2)-4p^3(\alpha+\beta)+2p^2}{(\alpha\beta-(\alpha+\beta)p+p^2)^4} \\
=\ &\frac{p^4-4p^2q+2q^2-4p(p^3-3pq)+6p^2(p^2-2q)-4p^4+2p^4}{(q-p^2+p^2)^4} \\
=\ &\frac{p^4-4p^2q+2q^2}{q^4}.
\end{aligned}$$

8. Let $y = \frac{(x+m)^2-4mn}{2(x-n)}$. Then $x^2 + 2(m-y) + m^2 - 4mn - 2ny = 0$. In order for $x$ to be real, we must have $4(m-y)^2 - 4(m^2 - 4mn + 2ny) \geq 0$. This is equivalent to $y^2 - 2(m+n)y + 4mn \geq 0$ or $(y-2m)(y-2n) \geq 0$. If $y$ lies between $2m$ and $2n$, one factor is positive but the other is negative. This is a contradiction. For any other value of $y$, there are two values of $x$, the roots of the quadratic equation in $x$ given earlier, which would yield that particular value of $y$.

10. Let $y = \frac{px^2+3x-4}{p+3x-4x^2}$. Then $(p+4y)x^2 + 3(1-y)x - (4+py) = 0$. In order for $x$ to be real, we must have $9(1-y)^2 + 4(p+4y)(4+py) \geq 0$. This is equivalent to

$$(16p+9)y^2 + 2(2p^2+23)y + (16p+9) \geq 0.$$

Now

$$\begin{aligned}
& (16p+9)((16p+9)y^2 + 2(2p^2+23)y + (16p+9)) \\
= \; & (16p+9)^2y^2 + 2(2p^2+23)(16p+9)y + (2p^2+23)^2 + \\
& (16p+9)^2 - (2p^2+23)^2 \\
= \; & ((16p+9)y + (2p^2+23))^2 + (2p^2+16p+32)(-2p^2+16p-14) \\
= \; & ((16p+9)y + (2p^2+23))^2 + 4(p+4)^2(p-1)(7-p).
\end{aligned}$$

If $1 \leq p \leq 7$, then $16p+9 > 0$ and $(p-1)(7-p) \geq 0$. The desired inequality follows.

12. Let $y = \frac{x^2-bc}{2x-b-c}$. Then $x^2 - 2yx + by + cy - bc = 0$. In order for $x$ to be real, we must have $4y^2 - 4(by+cy-bc) \geq 0$. This is equivalent to $(y-b)(y-c) \geq 0$. Hence $y$ cannot lie between $b$ and $c$.

14. Let $y = \frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$. Then $(ad-bcy)x^2 - (ac+bd)(1-y)x + (bc-ady) = 0$. In order for $x$ to be real, we must have $(ac+bd)^2(1-y)^2 - 4(ad-bcy)(bc-ady) \geq 0$. This is equivalent to $(ac-bd)^2y^2 - 2(a^2d^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2)y + (ac-bd)^2 \geq 0$. We may rewrite this expression as $((ac-bd)y - \frac{a^2c^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2}{ac-bd})^2 + (ac-bd)^2 - (\frac{a^2c^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2}{ac-bd})^2$. This is non-negative if so is $(a^2c^2-2abcd+b^2d^2)^2 - (a^2c^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2)^2$. This is a difference of two squares. Therefore, it is a product of two factors. One of them is $a^2c^2 - 2abcd + b^2d^2 - a^2c^2 - 2abcd - b^2d^2 + 2a^2d^2 + 2b^2c^2 = 2(ad-bc)^2 \geq 0$. The other factor is $a^2d^2 - 2abcd + b^2d^2 + a^2c^2 + 2abcd + b^2d^2 - 2a^2d^2 - 2b^2c^2 = 2(a^2-b^2)(c^2-d^2)$, which is positive if $a^2-b^2$ and $c^2-d^2$ have the same sign.

## Answers to Odd-numbered Exercises III

1. $-2 \leq n \leq 2$.
5. $bx^2 - 2ax + a = 0$.
11. $\frac{1}{3}$.

## Commentary III

Chapter III is a short chapter with only one topic. The determination of the possible values of rational functions is an application of the determinant; a topic that had been stressed in the previous chapter.

For the most part the students did not do well on this topic; the first major stumbling block in the course. The students were uncomfortable with inequalities and struggled with the use of two variables. I do not even think it was always clear what the meaning of the solution was. The students in our class, as is often the case with teachers, were primarily focused on individual manipulations. To solve these problems the student must do more than a sequence of moves, but rather make several logical connections also. The parameterized questions, such as Example 2, extend the situation to three variables. In these problems the discriminant is used twice, but in different ways.

This is a good section which offers a good application, challenge, and certainly fresh material. It would be better left until later in the course. We also found that the chapter disrupted the development of the theory of polynomial equations and it has been relocated to the section on inequalities.

Exercises 5, 6, 7, 9, 13 have been moved to the chapter on quadratics as they are on Vieta's Theorem and the discriminant.

# CHAPTER IV — MISCELLANEOUS EQUATIONS

In this chapter we propose to consider some miscellaneous equations; it will be seen that many of these can be solved by the ordinary rules for quadratic equations, but others require some special artifice for their solution.

**Example 1.**
Solve $8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63$.

**Solution:**

Let $y = x^{\frac{3}{2n}}$. Then $8y - \frac{8}{y} = 63$. Hence $0 = 8y^2 - 63y - 8 = (8y+1)(y-8)$, so that $y = -\frac{1}{8}$ or 8. When $y = -\frac{1}{8}$, $x = (-\frac{1}{2^3})^{\frac{2n}{3}} = \frac{1}{2^{2n}}$. When $y = 8$, $x = (2^3)^{\frac{2n}{3}} = 2^{2n}$.

**Example 2.**
Solve $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$.

**Solution:**

Let $y = \sqrt{\frac{x}{a}}$. Then $2y + \frac{3}{y} = \frac{b}{a} + \frac{6a}{b}$. Hence $0 = 2aby^2 - 6a^2y - b^2y + 3ab = (2ay-b)(by-3a)$, so that $y = \frac{b}{2a}$ or $\frac{3a}{b}$. When $y = \frac{b}{2a}$, $\frac{x}{a} = \frac{b^2}{4a^2}$ so that $x = \frac{b^2}{4a}$. When $y = \frac{3a}{b}$, $\frac{x}{a} = \frac{9b^2}{a^2}$ so that $x = \frac{9a^3}{b^2}$.

**Example 3.**
Solve $(x-5)(x-7)(x+6)(x+4) = 504$.

**Solution:**

We have $(x-5)(x+4) = x^2-x-20$ while $(x-7)(x+6) = x^2-x-42$. Let $y = x^2-x-20$. Then $y(y-22) = 504$. Hence $0 = y^2-22y-504 = (y+14)(y-36)$, so that $y = -14$ or 36. When $y = -14$, $0 = x^2 - x - 6 = (x+2)(x-3)$, so that $x = -2$ or 3. When $y = 36$, $0 = x^2 - x - 56 = (x+7)(x-8)$, so that $x = -7$ or 8.

Any equation which can be thrown into the form $ax^2+bx+c+p\sqrt{ax^2+bx+c} = q$ may be solved as follows. Putting $y = \sqrt{ax^2+bx+c}$, we obtain $y^2+py-q = 0$. Let $\alpha$ and $\beta$ be the roots of this equation. Then $\sqrt{ax^2+bx+c} = \alpha$ or $\sqrt{ax^2+bx+c} = \beta$. From these equations we shall obtain *four* values of $x$.

When no sign is prefixed to a radical, it is usually understood that it is to be taken as positive. Hence, if $\alpha$ and $\beta$ are both positive, all the four values of $x$ satisfy the original equation. If however $\alpha$ or $\beta$ is negative, the roots found from the resulting quadratic will satisfy the equation $ax^2+bx+c-p\sqrt{ax^2+bx+c} = q$ but not the original equation. Thus they are to be discarded.

**Example 4.**
Solve $x^2 - 5x + 2\sqrt{x^2-5x+3} = 12$.

**Solution:**

Let $y = \sqrt{x^2-5x+3}$. Then $y^2 - 3 + 2y = 12$. Hence $0 = y^2 + 2y - 15 =$

$(y+5)(y-3)$, so that $y=-5$ or 3. However, $\sqrt{x^2-5x+3} \geq 0$, so that we cannot have $y=-5$. When $y=3$, $0 = x^2-5x-6 = (x+1)(x-6)$, so that $x=-1$ or 6.

Before clearing an equation of radicals, it is advisable to examine whether any common factor can be removed by division.

**Example 5.**
Solve $\sqrt{x^2-7ax+10a^2} - \sqrt{x^2+ax-6a^2} = x-2a$.

**Solution:**

Factorization yields $\sqrt{(x-2a)(x-5a)} - \sqrt{(x-2a)(x+3a)} = x-2a$. Thus one root is $x=2a$. When $x \neq 2a$, we may divide throughout by $\sqrt{x-2a}$ and obtain $\sqrt{x-5a} - \sqrt{x+3a} = \sqrt{x-2a}$. Squaring both sides yields $x - 5a + x + 3a - 2\sqrt{(x-5a)(x+3a)} = x-2a$, which may be rewritten as $x = 2\sqrt{x^2-2ax-15a^2}$. Squaring both sides again yields $0 = 3x^2-8ax-60a^2 = (3x+10a)(x-6a)$, so that $x = -\frac{10a}{3}$ or $6a$. However, $x=6a$ is an extraneous root which does not satisfy the given equation. Hence the only other root is $x = -\frac{10a}{3}$.

The following artifice is sometimes useful.

**Example 6.**
Solve $\sqrt{x^2-4x+34} + \sqrt{3x^2-4x-11} = 9$.

**Solution:**

Note that $(3x^2-4x+34)-(3x^2-4x-11) = 45$. Dividing this by the given equation, making use of the factorization of the difference of two squares, we have $\sqrt{3x^2-4x+34} - \sqrt{3x^2-4x-11} = 5$. Now this equation is only true for the same values of $x$ which satisfies the given equation. Adding the two equations, we have $2\sqrt{3x^2-4x+34} = 14$. Hence $0 = 3x^2-4x-15 = (3x+5)(x-3)$, so that $x = -\frac{5}{3}$ or 3.

The solution of an equation of the form $ax^4 \pm bx^3 \pm cx^2 \pm bx + a = 0$, in which the coefficients of terms equidistant from the beginning and end are equal, can be made to depend on the solution of a quadratic. Equations of this type are known as reciprocal equations, and are so named because they are not altered when $x$ is changed into its reciprocal $\frac{1}{x}$. We shall revisit the notion of reciprocal equations in a later chapter.

**Example 7.**
Solve $12x^4-56x^3+89x^2-56x+12=0$.

**Solution:**

First, note that $x=0$ is not a root. Dividing throughout by $x^2$, $12(x^2+\frac{1}{x^2}) - 56(x+\frac{1}{x}) + 89 = 0$. Let $y = x+\frac{1}{x}$. Then $12(y^2-2) - 56y + 89 = 0$. Hence $0 = 12y - 56y - 65 = (6y-13)(2y-5)$, so that $y = \frac{13}{6}$ or $\frac{5}{2}$. When $y = \frac{13}{6} = x+\frac{1}{x}$, $0 = 6x^2-13x+6 = (3x-2)(2x-3)$, so that $x = \frac{2}{3}$ or $\frac{3}{2}$.

When $y = \frac{5}{2} = x + \frac{1}{x}$, $0 = 2x^2 - 5x + 2 = (2x-1)(x-2)$, so that $x = \frac{1}{2}$ or 2.

The following equation, though *not* reciprocal, may be solved in a similar manner.

**Example 8.**
Solve $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$.

**Solution:**

First, note that $x = 0$ is not a root. Dividing throughout by $x^2$, $6(x^2 + \frac{1}{x^2}) - 25(x - \frac{1}{x}) + 12 = 0$. Let $y = x - \frac{1}{x}$. Then $6(y^2+2) - 25y + 12 = 0$. Hence $0 = 6y^2 - 25y + 24 = (2y-3)(3y-8)$, so that $y = \frac{3}{2}$ or $\frac{8}{3}$. When $y = \frac{3}{2} = x - \frac{1}{x}$, $0 = 2x^2 - 3x - 2 = (2x+1)(x-2)$, so that $x = -\frac{1}{2}$ or 2. When $y = \frac{8}{3} = x - \frac{1}{x}$, $0 = 3x^2 - 8x - 3 = (3x+1)(x-3)$, so that $x = -\frac{1}{3}$ or 3.

When one root of a quadratic equation is obvious by inspection, the other root may often be readily obtained by making use of the properties of the roots of quadratic equations.

**Example 9.**
Solve $(1 - a^2)(x + a) - 2a(1 - x^2) = 0$.

**Solution:**

This is a quadratic, one of whose roots is clearly $a$. Also, since the equation may be rewritten as $2ax^2 + (1 - a^2)x - a(1 + a^2) = 0$, the product of the roots is $-\frac{1+a^2}{2}$. Hence the other root is $-\frac{1+a^2}{2a}$.

## EXERCISES

Solve the following equations:

1. $x^{-2} - 2x^{-1} = 8$.

2. $9 + x^{-4} = 10x^{-2}$.

3. $2\sqrt{x} + \frac{2}{\sqrt{x}} = 5$.

4. $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}$.

5. $x^{\frac{2}{n}} + 6 = 5x^{\frac{1}{n}}$.

6. $3x^{\frac{1}{2n}} - x^{\frac{1}{n}} - 2 = 0$.

7. $5\sqrt{\frac{3}{x}} + 7\sqrt{\frac{x}{3}} = 22\frac{2}{3}$.

8. $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$.

9. $6\sqrt{x} = 5x^{-\frac{1}{2}} - 13$.

10. $1 + 8x^{\frac{6}{5}} + 9x^{\frac{3}{5}} = 0$.

11. $3^{2x} + 9 = 10(3^x)$.

12. $5(5^x + 5^{-x}) = 26$.

13. $2^{2x+8} + 1 = 32(2^x)$.

14. $2^{2x+3} - 57 = 65(2^x - 1)$.

15. $\sqrt{2^x} + \frac{1}{\sqrt{2^x}} = 2.$ 16. $\frac{3}{\sqrt{2x}} - \frac{\sqrt{2x}}{5} = \frac{59}{10}.$

17. $(x-7)(x-3)(x+5)(x+1) = 1680.$

18. $(x+9)(x-3)(x-7)(x+5) = 385.$

19. $x(2x+1)(x-2)(2x-3) = 63.$

20. $(2x-7)(x^2-9)(2x+5) = 91.$

21. $x^2 + 2\sqrt{x^2+6x} = 24 - 6x.$

22. $3x^2 - 4x + \sqrt{3x^2-4x-6} = 18.$

23. $3x^2 - 7 + 3\sqrt{3x^2-16x+21} = 16x.$

24. $8 + 9\sqrt{(3x-1)(x-2)} = 3x^2 - 7x.$

25. $\frac{3x-2}{2} + \sqrt{2x^2-5x+3} = \frac{(x+1)^2}{3}.$

26. $7x - \frac{\sqrt{3x^2-8x+1}}{x} = (\frac{8}{\sqrt{x}} + \sqrt{x})^2.$

27. $\sqrt{4x^2-7x-15} - \sqrt{x^2-3x} = \sqrt{x^2-9}.$

28. $\sqrt{2x^2-9x+4} + 3\sqrt{2x-1} = \sqrt{2x^2+21x-11}.$

29. $\sqrt{2x^2+5x-7} + \sqrt{3(x^2-7x+6)} - \sqrt{7x^2-6x-1} = 0.$

30. $\sqrt{a^2+2ax-3x^2} - \sqrt{a^2+ax-6x^2} = \sqrt{2a^2+3ax-9x^2}.$

31. $\sqrt{2x^2+5x-2} - 2\sqrt{2x^2+5x-9} = 1.$

32. $\sqrt{3x^2-2x+9} + \sqrt{3x^2-2x-4} = 13.$

33. $\sqrt{2x^2-7x+1} - \sqrt{2x^2-9x+4} = 1.$

34. $\sqrt{3x^2-7x-30} - \sqrt{2x^2-7x-5} = x - 5.$

35. $x^4 + x^3 - 4x^2 + x + 1 = 0.$ 36. $x^4 + \frac{8}{9}x^2 + 1 = 3x^3 + 3x.$

37. $x^4 + 1 - 3(x^3 + x) = 2x^3.$ 38. $10(x^4+1) - 63x(x^2-1) + 52x^2 = 0.$

39. $\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}} = \frac{\sqrt{a}+1}{\sqrt{a}-1}.$ 40. $\frac{a+2x+\sqrt{a^2-4x^2}}{a+2x-\sqrt{a^2-4x^2}} = \frac{5x}{a}.$

41. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} - \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 8x\sqrt{x^2-3x+2}.$ 42. $\sqrt{x^2+x} + \frac{\sqrt{x-1}}{\sqrt{x^3-x}} = \frac{5}{2}.$

43. $\frac{x^3+1}{x^2-1} = x + \sqrt{\frac{6}{x}}.$ 44. $\frac{2^{x^2}}{2^{2x}} = \frac{8}{1}.$

45. $a^{2x}(a^2+1) = (a^{3x}+a^x)a$.

46. $\frac{8\sqrt{x-5}}{3x-7} = \frac{\sqrt{3x-7}}{x-5}$.

47. $\frac{18(7x-3)}{2x+1} = \frac{250\sqrt{2x+1}}{3\sqrt{7x-3}}$.

48. $(a+x)^{\frac{2}{3}} + 4(a-x)^{\frac{2}{3}} = 5(a^2-x^2)^{\frac{1}{3}}$.

49. $\sqrt{x^2+ax-1} - \sqrt{x^2+bx-1} = \sqrt{a} - \sqrt{b}$.

50. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} + \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 98$.

51. $x^4 - 2x^3 + x = 380$.

52. $27x^3 + 21x + 8 = 0$.

## Solutions to Even-numbered Exercises IV

2. Let $y = x^{-2}$. Then $9+y^2 = 10y$. Hence $0 = y^2-10y+9 = (y-1)(y-9)$, so that $y = 1$ or $9$. When $y = 1$, $x = \pm 1$. When $y = 9$, $x = \pm\frac{1}{3}$.

4. First, note that $x \neq 0$. Multiplying throughout by $x^{\frac{1}{4}}$, we have $6x = 7\sqrt{x} - 2$. Let $y = \sqrt{x}$. Then $6y^2 = 7y - 2$. Hence $0 = 6y^2 - 7y + 2 = (2y-1)(3y-2)$, so that $y = \frac{1}{2}$ or $\frac{2}{3}$. When $y = \frac{1}{2}$, $x = \frac{1}{4}$. When $y = \frac{2}{3}$, $x = \frac{4}{9}$.

6. Let $y = x^{\frac{1}{2n}}$. Then $3y - y^2 - 2 = 0$. Hence $0 = y^2-3y+2 = (y-1)(y-2)$, so that $y = 1$ or $2$. When $y = 1$, $x = 1$. When $y = 2$, $x = 2^{2n}$.

8. Let $y = \sqrt{\frac{x}{1-x}}$. Then $y + \frac{1}{y} = \frac{13}{6}$. Hence $0 = 6y^2 - 13y + 6 = (2y-3)(3y-2)$, so that $y = \frac{2}{3}$ or $\frac{3}{2}$. When $y = \frac{2}{3}$, $\frac{x}{1-x} = \frac{4}{9}$. Hence $9x = 4 - 4x$ so that $x = \frac{4}{13}$. When $y = \frac{3}{2}$, $\frac{x}{1-x} = \frac{9}{4}$. Hence $4x = 9 - 9x$ so that $x = \frac{9}{13}$.

10. Let $y = x^{\frac{3}{5}}$. Then $1 + 8y^2 + 9y = 0$. Hence $0 = 8y^2 + 9y + 1 = (8y+1)(y+1)$, so that $y = -\frac{1}{8}$ or $-1$. When $y = -\frac{1}{8}$, $x = -\frac{1}{32}$. When $y = -1$, $x = -1$.

12. Let $y = 5^x$. Then $5y + \frac{5}{y} = 26$. Hence $0 = 5y^2 - 26y + 5 = (5y-1)(y-5)$, so that $y = \frac{1}{5}$ or $5$. When $y = \frac{1}{5}$, $x = -1$. When $y = 5$, $x = 1$.

14. Let $y = 2^x$. Then $8y^2 - 57 = 65(y-1)$. Hence $0 = 8y^2 - 65y + 8 = (8y-1)(y-8)$, so that $y = \frac{1}{8}$ or $8$. When $y = \frac{1}{8}$, $x = -3$. When $y = 8$, $x = 3$.

16. Let $y = \sqrt{2x}$. Then $\frac{3}{y} - \frac{y}{5} = \frac{59}{10}$. Hence $0 = 2y^2 + 59y - 30 = (y+30)(2y-1)$, so that $y = -30$ or $\frac{1}{2}$. However, $\sqrt{2x} \geq 0$, so that $y \neq -30$. When $y = \frac{1}{2}$, $x = \frac{1}{8}$.

18. We have $(x+9)(x-7) = x^2+2x-63$ while $(x-3)(x+5) = x^2+2x-15$. Let $y = x^2+2x-15$. Then $y(y-48) = 385$. Hence $0 = y^2-48y+385 = (y+7)(y-55)$, so that $y = -7$ or 55. When $y = -7$, $0 = x^2+2x-8 = (x+4)(x-2)$, so that $x = -4$ or 2. When $y = 55$, $x^2+2x-70 = 0$ so that $x = \frac{-2\pm\sqrt{4+280}}{2} = -1 \pm \sqrt{71}$.

20. We have $(2x-7)(x+3) = 2x^2-x-21$ while $(2x+5)(x-3) = 2x^2-x-15$. Let $y = 2x^2-x-15$. Then $y(y-6) = 91$. Hence $0 = y^2-6y-91 = (y+7)(y-13)$, so that $y = -7$ or 13. When $y = -7$, $2x^2-x-8 = 0$, so that $x = \frac{1\pm\sqrt{1+64}}{4} = \frac{1\pm\sqrt{65}}{4}$. When $y = 13$, $0 - 2x^2-x-28 = (2x+7)(x-4)$, so that $x = -\frac{7}{2}$ or 4.

22. Let $y = \sqrt{3x^2-4x-6}$. Then $y^2+6+y = 18$. Hence $0 = y^2+y-12 = (y+4)(y-3)$, so that $y = -4$ or 3. However, $\sqrt{3x^2-4x-6} \geq 0$, so that we cannot have $y = -4$. When $y = 3$, $0 = 3x^2-4x-15 = (3x+5)(x-3)$, so that $x = -\frac{5}{3}$ or 3.

24. Let $y = \sqrt{3x^2-7x+2}$. Then $8+9y = y^2-2$. Hence $0 = y^2-9y-10 = (y+1)(y-10)$, so that $y = -1$ or 10. However, $\sqrt{3x^2-7x+2} \geq 0$, so that we cannot have $y = -1$. When $y = 10$, $0 = 3x^2-7x-98 = (3x+14)(x-7)$, so that $x = -\frac{14}{3}$ or 7.

26. First, note that $x \neq 0$. Multiplying throughout by $x$, we have $7x^2 + \sqrt{3x^2-8x+1} = 64+16x+x^2$, which may be rewritten as $6x^2-16x+\sqrt{3x^2-8x+1} = 64$. Let $y = \sqrt{3x^2-8x+1}$. Then $2y^2-2+y = 64$. Hence $0 = 2y^2+y-66 = (y+6)(2y-11)$, so that $y = -6$ or $\frac{11}{2}$. However, $\sqrt{3x^2-8x+1} \geq 0$, so that $y \neq -6$. When $y = \frac{11}{2}$, $12x^2-32x-117 = 0$, so that $x = \frac{32\pm\sqrt{1024+5616}}{24} = \frac{8\pm\sqrt{415}}{6}$.

28. Factorization yields $\sqrt{(2x-1)(x-4)} + 3\sqrt{2x-1} = \sqrt{(2x-1)(x+11)}$. Thus one root is $x = \frac{1}{2}$. When $x \neq \frac{1}{2}$, we may divide throughout by $\sqrt{2x-1}$ and obtain $\sqrt{x-4}+3 = \sqrt{x+11}$. Squaring both sides yields $x-4+9+6\sqrt{x-4} = x+11$, which may be rewritten as $\sqrt{x-4} = 1$. Hence $x-4 = 1$ and $x = 5$ is the other root.

30. Factorization yields
$\sqrt{(a+3x)(a-x)} - \sqrt{(a+3x)(a-2x)} = \sqrt{(a+3x)(2a-3x)}$. Thus one root is $x = -\frac{a}{3}$. When $x \neq -\frac{a}{3}$, we may divide throughout by $a+3x$. The resulting equation is $\sqrt{a-x} - \sqrt{a-2x} = \sqrt{2a-3x}$. Squaring yields $a-x+a-2x-2\sqrt{(a-x)(a-2x)} = 2a-3x$, which may be rewritten as $\sqrt{(a-x)(a-2x)} = 0$. Hence $(a-x)(a-2x) = 0$, so that $x = a$ and $x = \frac{a}{2}$. However, $x = a$ is an extraneous root which does not satisfy the given equation. Hence the only other root is $x = \frac{a}{2}$.

32. Note that $(3x^2-2x+9) - (3x^2-2x-4) = 13$. Dividing this by the given equation, we have $\sqrt{3x^2-2x+9} - \sqrt{3x^2-2x-4} = 1$. Adding

this to the given equation, we have $2\sqrt{3x^2 - 2x + 9} = 14$. Hence $0 = 3x^2 - 2x - 40 = (3x + 10)(x - 4)$, so that $x = -\frac{10}{3}$ or 4.

34. Squaring both sides yields $3x^2 - 7x - 30 = 2x^2 - 7x - 5 + x^2 - 10x + 25 + 2(x - 5)\sqrt{2x^2 - 7x - 5}$, which may be rewritten as $5(x - 5) = (x - 5)\sqrt{2x^2 - 7x - 5}$. Thus one root is $x = 5$. When $x \neq 5$, we may divide throughout by $x - 5$ and obtain $5 = \sqrt{2x^2 - 7x - 5}$. Hence $0 = 2x^2 - 7x - 30 = (2x + 5)(x - 6)$ and $x = -\frac{5}{2}$ and $x = 6$ are the other roots.

36. First, note that $x = 0$ is not a root. Dividing throughout by $x^2$, we have $x^2 + \frac{8}{9} + \frac{1}{x^2} = 3x + \frac{3}{x}$. Let $y = x + \frac{1}{x}$. Then $y^2 - 2 + \frac{8}{9} = 3y$. Hence $0 = 9y^2 - 27y - 10 = (3y + 1)(3y - 10)$, so that $y = -\frac{1}{3}$ or $\frac{10}{3}$. When $y = -\frac{1}{3} = x + \frac{1}{x}$, $3x^2 + x + 3 = 0$ so that $x = \frac{-1 \pm \sqrt{1 - 36}}{2} = \frac{-1 \pm \sqrt{35}i}{2}$. When $y = \frac{10}{3} = x + \frac{1}{x}$, $0 = 3x^2 - 10x + 3 = (3x - 1)(x - 3)$, so that $x = \frac{1}{3}$ or 3.

38. First, note that $x = 0$ is not a root. Dividing throughout by $x^2$, $10(x^2 + \frac{1}{x^2}) - 63(x - \frac{1}{x}) + 52 = 0$. Let $y = x - \frac{1}{x}$. Then $10(y^2 - 2) - 63y + 52 = 0$. Hence $0 = 10y^2 - 63y + 72 = (2y - 3)(5y - 24)$, so that $y = \frac{3}{2}$ or $\frac{24}{5}$. When $y = \frac{3}{2} = x - \frac{1}{x}$, $0 = 2x^2 - 3x - 2 = (2x + 1)(x - 2)$, so that $x = -\frac{1}{2}$ or 2. When $y = \frac{24}{5} = x - \frac{1}{x}$, $0 = 5x^2 - 24x - 5 = (5x + 1)(x - 5)$ so that $x = -\frac{1}{5}$ or 5.

40. Rationalizing the denominator, we have $\frac{a^2+4ax+4x^2+a^2-4x^2+2(a+2x)\sqrt{a^2-4x^2}}{a^2+4ax+4x^2-a^2+4x^2} = \frac{5x}{a}$, which may be rewritten as $\frac{(a+2x)(a+\sqrt{a^2-4x^2})}{2x(a+2x)} = \frac{5x}{a}$. Note that $x(a + 2x) \neq 0$. Hence $\frac{a+\sqrt{a^2-4x^2}}{2x} = \frac{5x}{a}$, which may be rewritten as $10x^2 - a^2 = a\sqrt{a^2 - 4x^2}$. Squaring yields $100x^2 - 20a^2x^2 + a^4 = a^4 - 4a^2x^2$. Since $x \neq 0$, we have $25x^2 = 4a^2$ so that $x = \pm\frac{2a}{5}$.

42. First, note that $x \neq 1$. Hence the equation may be rewritten as $\sqrt{x^2 + x} + \frac{1}{\sqrt{x^2+x}} = \frac{5}{2}$. Let $y = \sqrt{x^2 + x}$. Then $y + \frac{1}{y} = \frac{5}{2}$. Hence $0 = 2y^2 - 5y + 2 = (2y - 1)(y - 2)$, so that $y = \frac{1}{2}$ or 2. When $y = \frac{1}{2}$, $x^2 + x = \frac{1}{4}$. Hence $4x^2 + 4x - 1 = 0$, so that $x = \frac{-4 \pm \sqrt{16+16}}{8} = \frac{-1 \pm \sqrt{2}}{2}$. When $y = 2$, $x^2 + x = 4$. Hence $x^2 + x - 4 = 0$, so that $x = \frac{-1 \pm \sqrt{1+16}}{2} = \frac{-1 \pm \sqrt{17}}{2}$.

44. The given equation may be rewritten as $2^{x^2} == 2^{2x+3}$. Hence $0 = x^2 - 2x - 3 = (x + 1)(x - 3)$, so that $x = -1$ or 3.

46. The given equation may be rewritten as $8(x - 5)^{\frac{3}{2}} = (3x - 7)^{\frac{3}{2}}$. Squaring both sides yields $64(x - 5)^3 = (3x - 7)^3$ and taking cube roots yields $4(x - 5) = 3x - 7$, so that $x = 13$.

48. We have $0 = (a + x)^{\frac{2}{3}} - 5(a + x)^{\frac{1}{3}}(a - x)^{\frac{1}{3}} + 4(a - x)^{\frac{2}{3}} = ((a + x)^{\frac{1}{3}} - (a - x)^{\frac{1}{3}})((a + x)^{\frac{1}{3}} - 4(a - x)^{\frac{1}{3}})$. If the first factor is 0, then $a + x = a - x$ so that $x = 0$. If the second factor is 0, then $a + x = 64(a - x)$, so that $x = \frac{63a}{65}$.

50. Rationalizing the denominators, we have $x^2 + x^2 - 1 + 2x\sqrt{x^2 - 1} + x^2 + x^2 - 1 - 2x\sqrt{x^2 - 1} = 98$, which may be rewritten as $x^2 = 25$. Hence $x = \pm 5$.

52. By inspection, one of the roots is $x = -\frac{1}{3}$. When $x \neq -\frac{1}{3}$, we may divide throughout by $x + 3$ and obtain $9x^2 - 3x + 8 = 0$, so that the other roots are $x = \frac{3 \pm \sqrt{9-288}}{18} = \frac{1 \pm \sqrt{31}i}{6}$.

### Answers to Odd-numbered Exercises IV

1. $-\frac{1}{2}, \frac{1}{4}$.
3. $\frac{1}{4}, 4$.
5. $2^n, 3^n$,
7. $\frac{25}{147}, 27$.
9. $\frac{1}{9}, \frac{25}{4}$.
11. 0,2.
13. $-4$.
15. 0.
17. $-7, 9, 1 \pm 2\sqrt{6}i$.
19. $-\frac{3}{2}, 3, \frac{3 \pm \sqrt{47}i}{4}$.
21. $-8, 2$.
23. $\frac{1}{3}, 5$.
25. $\frac{1}{2}, 2, \frac{5 \pm \sqrt{201}}{4}$.
27. 1,3.
29. $-\frac{18}{5}, 1, 9$.
31. $-\frac{9}{2}, 2$.
33. 0,5.
35. $1, \frac{-3 \pm \sqrt{5}}{2}$.
37. $2 \pm \sqrt{3}, \frac{-1 \pm \sqrt{3}i}{2}$.
39. $-4a, 3a$.
41. 0,1,3.
43. $\frac{3}{2}$.
45. $\pm 1$.
47. 4.
49. $1, \frac{(\sqrt{a} - \sqrt{b})^2 + 4}{(\sqrt{a} + \sqrt{b})^2 - 4}$.
51. $-4, 5, \frac{1 \pm 5\sqrt{3}i}{2}$.

### Commentary IV

Miscellaneous Equations is a fun and rewarding chapter that allows students to extend their ability in quadratic equations to seemingly more difficult equations by the use of several tricks and substitutions. It is for sections like this one that Hall and Knight earned the reputation of solving highly contrived exercises in fortuitous situations. The principle omission in Hall and Knight's notes is a full disscussion of extraneous roots. In fact, many of the answers given in their text were extraneous.

Although of less theoretical weight, this is a good chapter for confidence and for stronger students may be the first time in the course that they solve an equation that they could not have solved before the course. Moreover, the strategies employed here are drawn upon later in the text.

## CHAPTER V — INEQUALITIES

**Definition** $[a > b]$

Any quantity $a$ is said to be greater than another quantity $b$ when $a - b$ is positive ; thus 2 is greater than $-3$, because $2 - (-3)$, or 5 is positive. Also $b$ is said to be less than $a$ when $b - a$ is negative; thus $-5$ is less than $-2$, because $-5 - (-2)$, or $-3$ is negative. In accordance with this definition, zero must be regarded as greater than any negative quantity.

**Theorem 1**

If $a > b$, $b > c$ are *real* numbers, and $d > 0$, then it is evident that

(i) $a + d > b + d$;
(ii) $a - d > b - d$;
(iii) $ad > bd$;
(iv) $\frac{a}{d} > \frac{b}{d}$;
(v) $a > c$ (Transitivity).

That is, an inequality will still hold after each side has been increased, decreased, multiplied, or divided by the same positive quantity. Moreover there is a logical connection between certain statements having a quantity in common.

**Proof of (i):**

Suppose $a > b$, then $a - b$ is positive. Then $a - b + d - d$ is positive, and therefore $a + d - (b + d)$ is positive. By definition then, $a + d > b + d$. The proofs of (ii)–(iv) are similar and left to the reader.

**Proof of (v):**

Since $a > b$ and $b > c$ we have $a - b$ and $b - c$ both positive. Since both are positive, their sum is positive hence $(a - b) + (b - c) = a - c$ is positive. So $a > c$.

**Example 1**

Suppose $y$ is positive, $x < y$, and $3y < z$, show that $x < z$.

**Solution:**

$3y - y = 2y > 0$ since $y$ is positive. Therefore $3y > y$. Then $x < y < 3y < z$ implies $x < z$ by transitivity.

The student should note that if each side is multiplied or divided by a negative quantity, the inequality must be reversed.

**Proof:**

Suppose $a > b$ and $d$ is positive. Then $a - b$ is positive and $b - a = (-a) - (-b)$ is negative, therefore $-a < -b$ and by (iii) above $-da < -db$. Hence $(-d)a < (-d)b$. For division, multiply by $\frac{1}{d}$.

**Theorem 2**
If $a_1 > b_1,\ a_2 > b_2,\ a_3 > b_3,\ \cdots,\ a_m > b_m$, all positive, it is clear that

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_m &> b_1 + b_2 + b_3 + \cdots + b_m \ ; \\ a_1 a_2 a_3 \cdots a_m &> b_1 b_2 b_3 \cdots b_m . \end{aligned}$$

A particular case of the later is if

$$\begin{aligned} a_1 = a_2 = \cdots = a_m &= a, \\ b_1 = b_2 = \cdots = b_m &= b, \end{aligned}$$

then we have $a^m > b^m$ for any positive integer $m$.

**Proof:**

Starting with $a_1 > b_1$ and adding $a_2$ we have

$$a_1 + a_2 > b_1 + a_2.$$

Since $a_2 > b_2$, by adding $b_1$ we have

$$a_2 + b_1 > b_1 + b_2.$$

By transitivity, Theorem 1(v), we have $a_1 + a_2 > b_1 + b_2$. Continuing in this way one can extend the result to $m$ terms on each side. The proof is similar for the second claim. The formal proof, by induction, of both of these claims is left as an exercise.

**Theorem 3 [Reciprocals]**
If $a > b$ and $a, b$ have the same sign, $\frac{1}{a} < \frac{1}{b}$. Conversely, if $b$ is negative and $a$ positive, then clearly the inequality need not be reversed.

**Proof:**

Let $a > b$, then $a - b$ is positive and $b - a$ is negative. Then $\frac{b-a}{ab} = \frac{1}{a} - \frac{1}{b}$ is negative (since $ab$ is positive) and therefore $\frac{1}{a} < \frac{1}{b}$.

**Theorem 4**
If $a > b > 0$, and if $p$, $q$ are positive integers then $\sqrt[q]{a} > \sqrt[q]{b}$ and therefore $a^{\frac{p}{q}} > b^{\frac{p}{q}}$.

**Proof: (by contradiction)**

Let $a > b$ and suppose that $\sqrt[q]{a} < \sqrt[q]{b}$. Raising both sides to the power of $q$ we have $a < b$, which is a contradiction. Therefore $\sqrt[q]{a} > \sqrt[q]{b}$, and by raising this to the power of $p$ we have $a^{\frac{p}{q}} > b^{\frac{p}{q}}$.

Using Theorems 3 and 4 we may now handle inequalities containing any rational exponents.

## AM-GM Inequality

**Theorem [AM-GM for n = 2]**
For $a$, $b$ positive unequal numbers we have, $\frac{a+b}{2} > \sqrt{ab}$.

**Proof:**

The square of every real quantity is positive, and therefore greater than zero. Thus $(\sqrt{a} - \sqrt{b})^2$ is positive;

$$\begin{aligned} a - 2\sqrt{a}\sqrt{b} + b &> 0; \\ a + b &> 2\sqrt{ab}; \\ \frac{a+b}{2} &> \sqrt{ab}. \end{aligned}$$

That is, the arithmetic mean of two positive quantities is greater than their geometric mean. The inequality becomes an equality when the quantities are equal.

**Theorem [AM-GM for general n]**
Let $x_1, x_2, \cdots, x_n$ be positive quantities. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}.$$

**Proof**

The proof is by a special induction. We have shown the case $n = 2$. We will show that if the claim holds for $n = k$ then the case $n = 2k$ holds too. Next we will show that the $n = k + 1$ case implies the $n = k$ case. Thus starting from the basis case $n = 2$ we shall imply $n = 4, 3, 6, 5, 8, 7, 10, 9 \ldots$ by doubling and back-stepping.

Suppose that the claim holds for $n = k$. Then consider;

$$\begin{aligned} \frac{x_1 + x_2 + \cdots + x_k + x_{k+1} + \cdots + x_{2k}}{2k} &\geq \frac{\sqrt[k]{x_1 \cdot x_2 \cdots x_k} + \sqrt[k]{x_{k+1} \cdot x_{k+2} \cdots x_{2k}}}{2} \\ &> \sqrt[2k]{x_1 \cdot x_2 \cdots x_k \cdot x_{k+1} \cdot x_{k+2} \cdots x_{2k}} \end{aligned}$$

Suppose now that the claim holds for $n = k + 1$ and consider;

$$\begin{aligned} a = \frac{x_1 + x_2 + \cdots + x_k}{k} &= \frac{x_1 + x_2 + \cdots + x_k + a}{k+1} \\ &\geq \sqrt[k+1]{(x_1 \cdot x_2 \cdots x_k) \cdot a} \end{aligned}$$

Now, raising both sides to the power of $k + 1$, we have

$$\begin{aligned} a^{k+1} &\geq (x_1 \cdot x_2 \cdots x_k) \cdot a \\ a^k &\geq x_1 \cdot x_2 \cdots x_k. \end{aligned}$$

So,

$$\frac{x_1 + x_2 + \cdots + x_k}{k} \geq \sqrt[k]{x_1 \cdot x_2 \cdots x_k}.$$

The theorem is thus proved for all $n$. The student should note aditionally that it is clear that equality is attained when all $x$'s are equal.

The AM-GM inequality will be found very useful, especially in the case of inequalities in which the letters are involved symmetrically.

**Example 2**
If $a, b, c$ denote positive quantities, prove that

$$a^2 + b^2 + c^2 > bc + ca + ab;$$

$$2(a^3 + b^3 + c^3) > bc(b + c) + ca(c + a) + ab(a + b).$$

**Solution:**

Using the AM-GM inequality, we have

$$\begin{array}{rcl} b^2 + c^2 & > & 2bc; \\ c^2 + a^2 & > & 2ca; \\ a^2 + b^2 & > & 2ab; \\ \text{whence by addition, } a^2 + b^2 + c^2 & > & bc + ca + ab. \end{array}$$

This result is in fact true for any real $a, b, c$. From the first line above we have

$$\begin{array}{rrcl} & b^2 - bc + c^2 & > & bc; \\ \text{so,} & b^3 + c^3 & > & bc(b + c). \end{array}$$

By writing down similar inequalities and adding we obtain

$$2(a^3 + b^3 + c^3) > bc(b + c) + ca(c + a) + ab(a + b)$$

It should be noted that since we multiply by factors such as $(b + c)$ above we do require that they be positive in order that the inequality need not be reversed.

**Example 3**
If $x$ may have any real value find which is the greater, $x^3 + 1$ or $x^2 + x$.

**Solution:**

$$\begin{aligned} x^3 + 1 - (x^2 + x) &= x^3 - x^2 - (x - 1) \\ &= (x^2 - 1)(x - 1) \\ &= (x - 1)^2(x + 1). \end{aligned}$$

Now $(x-1)^2$ is positive, hence the inequality depends only on whether $(x+1)$ is positive or negative. That is,

$$\begin{aligned} x^3+1 &> (x^2+x) \quad \text{iff} \quad x > -1 \\ x^3+1 &< (x^2+x) \quad \text{iff} \quad x < -1 \\ x^3+1 &= (x^2+x) \quad \text{iff} \quad x = -1. \end{aligned}$$

**Theorem [Constrained Optima]**
Let $a$ and $b$ be two positive quantities, $S$ their sum and $P$ their product; then from the identity

$$4ab = (a+b)^2 - (a-b)^2,$$

we have

$$4P = S^2 - (a-b)^2, \text{ and } S^2 = 4P + (a-b)^2.$$

Hence, if $S$ is given, $P$ is greatest when $a=b$ and if $P$ is given, $S$ is least when $a=b$.

That is, *if the sum of two positive quantities is given, their product is greatest when they are equal; and if product of two positive quantities is given, their sum is least when they are equal.*

**Application**
*To find the greatest value of a product the sum of whose factors is constant.*

Let there be $n$ factors $a, b, c, \cdots, k$, and suppose that their sum is constant and equal to $s$.

Consider the product $abc\cdots k$, and suppose that $a$ and $b$ are any two unequal factors. If we replace the two unequal factors $a$, $b$ by the two equal factors $\frac{a+b}{2}$ and $\frac{a+b}{2}$ the product is increased while the sum remains unaltered; hence so long as the product contains two unequal factors it can be increased without altering the sum of the factors; therefore the product is greatest when all the factors are equal. In this case the value of each of the $n$ factors is $\frac{s}{n}$, and the greatest value of the product is $\left(\frac{s}{n}\right)^n$, or

$$\left(\frac{a+b+c+\cdots+k}{n}\right)^n.$$

**Corollary**
If $a, b, c, \cdots k$ are unequal,

$$\left(\frac{a+b+c+\cdots+k}{n}\right)^n > abc\cdots k;$$

that is

$$\frac{a+b+c+\cdots+k}{n} > (abc\cdots k)^{\frac{1}{n}}.$$

This result is precisely the AM-GM inequality of the previous section.

**Example 4**
Show that $(1^r + 2^r + 3^r + \cdots + n^r)^n > n^n(n!)^r$; where $r$ is any real quantity.

**Solution:**

Since

$$\begin{aligned} \frac{1^r + 2^r + 3^r + \cdots + n^r}{n} &> (1^r 2^r 3^r \cdots n^r)^{\frac{1}{n}}; \\ \left(\frac{1^r + 2^r + 3^r + \cdots + n^r}{n}\right)^n &> 1^r 2^r 3^r \cdots n^r; \\ &> (n!)^r. \end{aligned}$$

### Application

*To find the greatest value of $a^m b^n c^p \cdots$ when $a+b+c+\cdots$ is constant; $m, n, p, \cdots$ being positive integers.*

Since $m, n, p, \cdots$ are constants, the expression $a^m b^n c^p \cdots$ will be greatest when $\left(\frac{a}{m}\right)^m \left(\frac{b}{n}\right)^n \left(\frac{c}{p}\right)^p \cdots$ is greatest. But this last expression is the product of $m+n+p+\cdots$ factors whose sum is $m\left(\frac{a}{m}\right)+n\left(\frac{b}{n}\right)+p\left(\frac{c}{p}\right)+\cdots$, or $a+b+c+\cdots$, and therefore constant. Hence $a^m b^n c^p \cdots$ will be greatest when the factors

$$\frac{a}{m}, \frac{b}{n}, \frac{c}{p}, \cdots$$

are all equal, that is when

$$\frac{a}{m} = \frac{b}{n} = \frac{c}{p} = \frac{a+b+c+\cdots}{m+n+p+\cdots}.$$

Thus the greatest value is

$$m^m n^n p^p \cdots \left(\frac{a+b+c+\cdots}{m+n+p+\cdots}\right)^{m+n+p+\cdots}$$

**Example 5**
Find the greatest value of $(a+x)^3(a-x)^4$ for any real value of $x$ numerically less than $a$.

**Solution:**

The given expression is greatest when $\left(\frac{a+x}{3}\right)^3 \left(\frac{a-x}{4}\right)^4$ is greatest; but the sum of the factors of this expression is $3\left(\frac{a+x}{3}\right) + 4\left(\frac{a-x}{4}\right) = 2a$; hence $(a+x)^3(a-x)^4$ is greatest when $\left(\frac{a+x}{3}\right) = \left(\frac{a-x}{4}\right)$, or $x = -\frac{a}{7}$.

Thus the greatest value is $\frac{6^3 \cdot 8^4}{7^7} a^7$.

The determination of maximum and minimum values may often be more simply effected by the solution of a quadratic equation than by the foregoing methods. Instances of this have already occurred; we add a further example.

**Example 6**

Divide an odd integer into two integral parts whose product is a maximum.

**Solution:**

Denote the integer by $2n+1$; the two parts by $x$ and $2n+1-x$; and the product by $y$, then $(2n+1)x-x^2=y$ ,whence, by the quadratic formula,

$$2x=(2n+1)\pm\sqrt{(2n+1)^2-4y};$$

but the quantity under the radical must be positive, and therefore $y$ cannot be greater than $\frac{1}{4}(2n+1)^2$, or $n^2+n+\frac{1}{4}$; and since $y$ is integral its greatest value must be $n^2+n$; in which case $x=n+1$, or $n$; thus the two parts are $n$ and $n+1$.

**Example 7**

Find the minimum value of $\frac{(a+x)(b+x)}{c+x}$.

**Solution:**

Put $c+x=y$; then

$$\begin{aligned}\frac{(a+x)(b+x)}{c+x} &= \frac{(a-c+y)(b-c+y)}{y};\\ &= \frac{(a-c)(b-c)}{y}+y+a-c+b-c;\\ &= \left(\frac{\sqrt{(a-c)(b-c)}}{\sqrt{y}}-\sqrt{y}\right)^2+a-c+b-c+2\sqrt{(a-c)(b-c)}.\end{aligned}$$

Hence the expression is a minimum when the square term is zero; that is when $y=\sqrt{(a-c)(b-c)}$.

Thus the minimum value is

$$a-c+b-c+2\sqrt{(a-c)(b-c)};$$

and the corresponding value of $x$ is $\sqrt{(a-c)(b-c)}-c$.

## EXERCISES V

1. Prove that $(ab+xy)(ax+by)\geq 4abxy$.
2. Prove that $(b+c)(c+a)(a+b)\geq 8abc$.
3. Show that the sum of any real positive quantity and its reciprocal is never less than 2.
4. If $a^2+b^2=1$, and $x^2+y^2=1$, show that $ax+by\leq 1$.
5. If $a^2+b^2+c^2=1$, and $x^2+y^2+z^2=1$, show that $ax+by+cz\leq 1$.

6. If $a \geq b$, show that $a^a b^b \geq a^b b^a$, and $\frac{b}{a} \leq \frac{1+b}{1+a}$.

7. Show that $(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq 9x^2y^2z^2$.

8. Find which is the greater $3ab^2$ or $a^3 + 2b^3$.

9. Prove that $a^3b + ab^3 \leq a^4 + b^4$.

10. Prove that $6abc \leq bc(b+c) + ca(c+a) + ab(a+b)$.

11. Show that $b^2c^2 + c^2a^2 + a^2b^2 \geq abc(a+b+c)$.

12. Which is the greater $x^3$ or $x^2 + x + 2$ for positive values of $x$?

13. Show that $x^3 + 13a^2x \geq 5ax^2 + 9a^3$, if $x > a$.

14. Find the greatest value of $x$ in order that $7x^2 + 11$ may be greater than $x^3 + 17x$.

15. Find the minimum value of $x^2 - 12x + 40$, and the maximum value of $24x - 8 - 9x^2$.

16. Show that $(n!)^2 > n^n$, and $2 \cdot 4 \cdot 6 \cdots 2n < (n+1)^n$.

17. Show that $(x+y+z)^3 \geq 27xyz$.

18. Show that $n^n > 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

19. If $n$ be a positive integer greater than 2, show that $2^n > 1 + (\sqrt{2})^{n-1}$.

20. Show that $(n!)^3 < n^n \left(\frac{n+1}{2}\right)^{2n}$.

21. Show that

$$\begin{aligned} &(1) \quad (x+y+z)^3 \geq 27(y+z-x)(z+x-y)(x+y-z). \\ &(2) \quad xyz \geq (y+z-x)(z+x-y)(x+y-z). \end{aligned}$$

22. Find the maximum value of $(7-x)^4(2+x)^5$ when $x$ lies between 7 and -2.

23. Find the minimum value of $\frac{(5+x)(2+x)}{1+x}$.

## Solutions to Exercises V

2. We have $\frac{b+c}{2} \geq \sqrt{bc}$ so that $b + c \geq 2\sqrt{bc}$. Similarly, $c + a \geq 2\sqrt{ca}$ and $a + b \geq 2\sqrt{ab}$. Hence $(b+c)(c+a)(a+b) \geq 8abc$.

4. We have $a^2 + x^2 \geq 2ax$ and $b^2 + y^2 \geq 2by$. Thus, $a^2 + b^2 + x^2 + y^2 \geq 2(ax + by)$. Hence, $1 \geq ax + by$.

6. We have $a^ab^b - a^bb^a = (ab)^b(a^{a-b} - b^{a-b}) > 0$. Also, $b(1+a) = b + ab < a + ab = a(1+b)$ so that $\frac{b}{a} < \frac{1+b}{1+a}$.

8. We have $a^3 + 2b^3 = a^3 + b^3 + b^3 \geq 3(\sqrt[3]{a^3b^3b^3}) = 3ab^2$.

10. We have

$$\begin{aligned} bc(b+c)+ca(c+a)+ab(a+b) &= c(b^2+a^2)+b(c^2+a^2)+a(b^2+c^2) \\ &\geq 2c\sqrt{a^2b^2}+2b\sqrt{c^2a^2}+2a\sqrt{b^2c^2} \\ &= 6abc. \end{aligned}$$

12. We have $x^3-x^2-x-2=(x-2)(x^2+x+1)$. For $x>0$, $x^2+x+1>0$. Hence $x^3>x^2+x+2$ if and only if $x>2$.

14. We have

$$\begin{aligned} x^3-7x^2+17x-11 &= (x-1)(x^2-8x+11) \\ &= (x-1)((x-4)^2-5) \\ &= (x-1)(x-4+\sqrt{5})(x-4-\sqrt{5}). \end{aligned}$$

For $x>1$, all three factors are positive. It follows that the maximum value of $x$ for which $7x^2+11 \geq x^3+17x$ is $x=1$.

16. We have $n=1(n)\leq 2(n-1)\leq 3(n-2)\leq\cdots$. Hence $n^2<(n!)^2$ since not all equalities can hold simultaneously. On the other hand, $2(2n)<4(2n-2)<6(2n-4)<\cdots<(n+1)^2$. Hence $2\times4\times6\times\cdots\times2n<(n+1)^n$.

18. We have $1(2n-1)<3(2n-3)<5(2n-5)<\cdots<n^2$. Hence $1\times3\times\cdots\times(2n-1)<n^n$.

20. We have $1(n)<2(n-1)<3(n-2)<\cdots<(\frac{n+1}{2})^2$. Hence $(n!)^2<(\frac{n+1}{2})^{2n}$. Since $n!<n^n$, we have $(n!)^3<n^n(\frac{n+1}{2})^{2n}$.

22. The expression $(7-x)^4(2+x)^5$ takes on its maximum value at the same time as the expression $(35-5x)^4(8+4x)^5$. In the latter expression, the sum of the nine factors is 180. It follows that $35-5x=20$ so that $x=3$. Hence the maximum value of the original expression is $4^45^5$.

## Answers to Exercises V

8. $a^3+2b^3$ is the greater.
12. $x^3>$ or $<x^2+x+2$ according as $x>$ or $<2$.
14. The greatest value of $x$ is 1.
15. 4; 8.
22. $4^4\cdot5^5$; when $x=3$.
23. 9, when $x=1$.

## Commentary V

The inequalities chapter is a separate topic from most of the text and was placed in this position in the sequence for convenience. In Part 2, Inequalities is placed after the completion of the quadratic/cubic/quartic sequence.

Hall and Knight give a less thorough development of the basic properties of Theorem 1. This development is important and not too difficult. The careful analysis of the behavior of a new component mirrors the analysis granted to imaginary numbers after their introduction in Chapter II. It is a theme in abstract algebra to introduce an object and some operations on it and to "see what happens". These sections illustrate that model.

The Arithmetic-Geometric mean inequality is the main result of the section. The proof for $n = 2$ and some applications in the exercises were eventually taken quite well by the students. The general proof by induction was, not surprisingly, difficult, especially for those who have not seen induction. It must be remembered, however, that this is not the sort of topic on which the students in a class like this should be tested. The proof is something to see and to understand. It is notable that, as in the case of extending the quadratic formula, it is easier to go from $n = 2$ to $n = 4$ than it is to go from $n = 2$ to $n = 3$.

# CHAPTER VI — THE THEORY OF EQUATIONS

In Chapter II, we established certain relations between the roots and the coefficients of quadratic equations. We shall now investigate similar relations which hold in the case of equations of the $n^{th}$ degree, and some properties in the general theory of equations.

Let $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$ be a polynomial in $x$ of degree $n$. Then $f(x) = 0$ is the general polynomial equation of degree $n$. Dividing throughout if necessary by $p_0 \neq 0$, we see that without any loss of generality we may take $p_0 = 1$. Unless otherwise stated, the coefficients $p_1, p_2, \ldots, p_n$ will always be supposed rational. Any value of $x$ which makes $f(x)$ vanish is called a root of the equation $f(x) = 0$.

**Remainder Theorem.**
When a polynomial $f(x)$ in $x$ is divided by $x - \alpha$, the remainder is $f(\alpha)$.

**Proof:**

Since $f(x)$ is divided by a linear polynomial $x - \alpha$, the remainder does not involve $x$. Let $Q(x)$ be the quotient and $R$ the remainder. Then $f(x) = (x-\alpha)Q(x)+R$. Since $R$ does not involve $x$, it will remain unaltered whatever value we give to $x$. Put $x = \alpha$. Then $f(\alpha) = (\alpha - \alpha)Q(\alpha) + R = R$. Since $Q(x)$ is finite for finite values of $x$, $R = f(\alpha)$.

**Factor Theorem.**
If $f(\alpha) = 0$, then $f(x)$ is divisible by $x - \alpha$, and conversely.

**Proof:**

By the Remainder Theorem, both conditions are equivalent to $R = 0$.

**Fundamental Theorem of Algebra.**
Every polynomial equation with complex coefficients has a root, real or imaginary.

The proof of this important result is beyond the scope of the present work. From this, it can be deduced that a polynomial equation of degree $n$ has exactly $n$ roots. Denote the given equation by $f(x) = 0$. Then $f(x) = 0$ has a root $\alpha_1$ and $f(x)$ is divisible by $x - \alpha_1$, so that $f(x) = (x - \alpha_1)f_1(x)$ for some polynomial $f_1(x)$ of degree $n - 1$. Again, the equation $f_1(x) = 0$ has a root $\alpha_2$. Then $f_1(x)$ is divisible by $x - \alpha_2$, so that $f_1(x) = (x - \alpha_2)f_2(x)$ for some polynomial $f_2(x)$ of degree $n - 2$. Proceeding in this way, we obtain $f(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n)$. Hence the equation $f(x) = 0$ bas $n$ roots, since $f(x)$ vanishes when $x$ has any of the values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

Also the equation cannot have more than $n$ roots; for if $x$ has any value different from any of the quantities $\alpha_1, \alpha_2, \ldots, \alpha_n$, all the factors on the right are different from zero, and therefore $f(x)$ cannot vanish for that value of $x$.

In the above investigation some of the quantities $\alpha_1, \alpha_2, \ldots, \alpha_n$ may be equal. In this case, however, we shall suppose that the equation still has $n$ roots, although these are not all different.

We now investigate the relations between the roots and the coefficients in any equation. Let us denote the equation by $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$, and the roots by $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. Then we have identically

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_1x + p_0 = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\cdots(x-\alpha_n).$$

Equating coefficients of like powers of $x$ in this identity,

$$\begin{aligned} -p_1 &= \text{sum of the roots;} \\ p_2 &= \text{sum of the products of the roots taken two at a time;} \\ -p_3 &= \text{sum of the products of the roots taken three at a time;} \\ \cdots &= \cdots; \\ (-1)^n p_n &= \text{product of the roots.} \end{aligned}$$

If the coefficient of $x^n$ is $p_0$, then on dividing each term by $p_0$, the equation becomes $x^n+\frac{p_1}{p_0}x^{n-1}+\frac{p_2}{p_0}x^{n-2}+\cdots+\frac{p_{n-1}}{p_0}x+\frac{p_n}{p_0}$ and, with the notation in Chapter V, we have $\sum\alpha_i = -\frac{p_1}{p_0}$, $\sum\alpha_i\alpha_j = \frac{p_2}{p_0}$, $\sum\alpha_i\alpha_j\alpha_k = -\frac{p_3}{p_0}$, $\cdots$, $\alpha_1\alpha_2\alpha_3\cdots\alpha_n = (-1)^n\frac{p_n}{p_0}$.

**Example 1.**

If $\alpha, \beta$ and $\gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$, form the equation whose roots are $\alpha^2, \beta^2$ and $\gamma^2$.

**Solution:**

The required equation is $(y-\alpha^2)(y-\beta^2)(y-\gamma^2) = 0$, or $(x^2-\alpha^2)(x^2-\beta^2)(x^2-\gamma^2) = 0$ if $y = x^2$. In other words, $(x-\alpha)(x-\beta)(x-\gamma)(x+\alpha)(x+\beta)(x+\gamma) = 0$. However, $(x-\alpha)(x-\beta)(x-\gamma) = x^3 + px^2 + qx + r$. Hence $(x+\alpha)(x+\beta)(z+\gamma) = x^3 - px^2 + qx - r$. Thus the required equation is

$$\begin{aligned} 0 &= (x^3 + px^2 + qx + r)(x^3 - px^2 + qx - r) \\ &= (x^3 + qx)^2 - (px^2 + r)^2 \\ &= x^6 + (2q - p^2)x^4 + (q^2 - 2pr)x^2 - r^2. \end{aligned}$$

If we replace $x^2$ by $y$, we obtain $y^3 + (2q - p^2)y^2 + (q^2 - 2pr)y - r^2 = 0$.

The student might suppose that the relations established so far would enable him or her to solve any proposed equation; for the number of the relations is equal to the number of the roots. A little reflection will show that this is not the case. Suppose we eliminate any $n-1$ of the roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ in order to obtain an equation to determine the remaining one. Since these roots are involved symmetrically in each of the equations, it is clear that we shall always obtain an equation having the same coefficients; this equation is therefore the original equation with some one of the roots $\alpha_k$ substituted for $x$.

Let us take for example the equation $x^3 + px^2 + qx + r = 0$. Let $\alpha, \beta$ and $\gamma$ be the roots. Then

$$\begin{aligned} \alpha + \beta + \gamma &= -p, \\ \alpha\beta + \beta\gamma + \gamma\alpha &= q, \\ \alpha\beta\gamma &= -r. \end{aligned}$$

Multiply these equations by $\alpha^2, -\alpha$ and 1 respectively and add, we obtain $\alpha^3 = -p\alpha^2 - q\alpha - r$, that is, $\alpha^3 + p\alpha^2 + q\alpha + r = 0$, which is the original equation with $\alpha$ in the place of $x$.

The above process of elimination is quite general, and is applicable to equations of any degree.

If two or more of the roots of an equation are connected by an assigned relation, the properties proved earlier will sometimes enable us to obtain the complete solution.

**Example 2.**
Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in arithmetical progression.

**Solution:**

Denote the roots by $\alpha - d, \alpha$ and $\alpha + d$. Then the sum of the roots is $3\alpha$, the sum of the products of the roots two at a time is $3\alpha^2 - d^2$, and the product of the roots is $\alpha(\alpha^2 - d^2)$. Hence we have the equations $3\alpha = 6, 3\alpha^2 - d^2 = \frac{23}{4}$ and $\alpha(\alpha^2 - d^2) = -\frac{9}{4}$. From the first equation, we find $\alpha = 2$, and from the second $d = \pm\frac{5}{2}$. Since these values satisfy the third, the three equations are consistent. Thus the roots are $-\frac{1}{2}$, 2 and $\frac{9}{2}$.

**Example 3.**
Solve the equation $24x^3 - 14x^2 - 63x + 45 = 0$, one root being double another.

**Solution:**

Denote the roots by $\alpha, 2\alpha$ and $\beta$. Then we have $3\alpha + \beta = \frac{7}{12}, 2\alpha^2 + 3\alpha\beta = -\frac{21}{8}$ and $2\alpha^2 b = -\frac{15}{8}$. From the first two equations, we obtain $0 = 8\alpha^2 - 2\alpha - 3 = (4\alpha - 3)(2\alpha + 1)$, so that $\alpha = \frac{3}{4}$ or $-\frac{1}{2}$. However, when $\alpha = -\frac{1}{2}$, $\beta = \frac{25}{12}$, but these values do not satisfy the third equation. Hence the roots are $\alpha = \frac{3}{4}, 2\alpha = \frac{3}{2}$ and $\beta = -\frac{5}{3}$.

Although we may not be able to find the roots of an equation, we can make use of the relations proved earlier to determine the values of symmetrical functions of the roots.

**Example 4.**
Find the sum of the squares and of the cubes of the roots of the equation $x^3 - px^2 + qx - r = 0$.

**Solution:**

Denote the roots by $\alpha, \beta$ and $\gamma$. Then $\alpha+\beta+\gamma = p$ and $\beta\gamma+\gamma\alpha+\alpha\beta = q$. Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\beta\gamma+\gamma\alpha+\alpha\beta) = p^2-2q$. Substitute $\alpha, \beta$ and $\gamma$ for $x$ in the given equation. Addition yields $\alpha^3+\beta^3+\gamma^3 = -p(\alpha^2+\beta^2+\gamma^2)+q(\alpha+\beta+\gamma)-3r = 0$. Hence $\alpha^3+\beta^3+\gamma^3 = p(p^2-2q)-pq+3r = p^3-3pq+3r$.

**Example 5.**

If $\alpha, \beta, \gamma$ and $\delta$ are the roots of $x^4+px^3+qx^2+rx+s=0$, find the value of $\sum \alpha^2\beta$.

**Solution:**

We have $\alpha+\beta+\gamma+\delta = -p, \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta-\gamma\delta = q$ and $\alpha\beta\gamma+\alpha\beta\delta+\alpha\beta\gamma\delta+\beta\gamma\delta = -r$. From these equations we have $-pq = \sum\alpha^2\beta+3\sum\alpha\beta\gamma = \sum\alpha^2\beta-3r$. Hence $\sum\alpha^2\beta = 3r-pq$.

## EXERCISES VI

Form the equation whose roots are:

1. $\frac{2}{3}, \frac{3}{2}, \pm\sqrt{3}$.
2. $0, 0, 2, 2, -3, -3$.
3. $2, 2, -2, -2, 0, 5$.
4. $\alpha+\beta, \alpha-\beta, -\alpha+\beta, -\alpha-\beta$.

Solve the equations:

5. $x^4-16x^3+86x^2-176x+105=0$, two roots being 1 and 7.
6. $4x^3+16x^2-9x-36=0$, the sum of two of the roots being zero.
7. $4x^3+20x^2-23x+6=0$, two of the roots being equal.
8. $3x^3-26x^2+52x-24=0$, the roots being in geometrical progression.
9. $2x^3-x^2-22x-24=0$, two of the roots being in the ratio of 3:4.
10. $24x^3+46x^2+9x-9=0$, one root being double another of the roots.
11. $8x^4-2x^3-27x^2+6x+9=0$, two of the roots being equal but opposite in sign.
12. $54x^3-39x^2-26x+16=0$, the roots being in geometrical progression.
13. $32x^3-48x^2+22x-3=0$, the roots being in arithmetical progression.
14. $6x^4-29x^3+40x^2-7x-12=0$, the product of two of the roots being 2.
15. $x^4-2x^3-21x^2+22x+40=0$, the roots being in arithmetical progression.

16. $27x^4 - 195x^3 + 494x^2 - 520x + 192 = 0$, the roots being in geometrical progression.

17. $18x^3 + 81x^2 + 121x + 60 = 0$, one root being half the sum of the other two.

18. If $\alpha, \beta$ and $\gamma$ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the value of

    (a) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$;

    (b) $\frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2} + \frac{1}{\alpha^2\beta^2}$.

19. If $\alpha, \beta$ and $\gamma$ are the roots of $x^3 + qx + r = 0$, find the value of

    (a) $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$;

    (b) $(\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1} + (\alpha + \beta)^{-1}$.

20. Find the sum of the squares and of the cubes of the roots of $x^4 + qx^2 + rx + 8 = 0$.

21. Find the sum of the fourth powers of the roots of $x^3 + qx + r = 0$.

## Solutions to Even-numbered Exercises VI

2. We have $0 = x^2(x-2)^2(x+3)^2 = x^2(x^2 - 4x + 4)(x^2 + 6x + 9) = x^6 + 2x^5 - 11x^4 - 12x^3 + 36x^2$.

4. We have

$$\begin{aligned} 0 &= (x - (\alpha + \beta))(x - (\alpha - \beta))(x - (-\alpha + \beta))(x - (-\alpha - \beta)) \\ &= ((x-\alpha)^2 - \beta^2)((x+\alpha)^2 - \beta^2)) \\ &= (x-\alpha)^2(x+\alpha)^2 - \beta^2((x-\alpha)^2 + (x+\alpha)^2) + \beta^4 \\ &= x^4 - 2\alpha^2x^2 + \alpha^4 - 2\beta^2x^2 - 2\alpha^2\beta^2 + \beta^4 \\ &= x^4 - 2(\alpha^2 + \beta^2) + (\alpha^2 - \beta^2)^2. \end{aligned}$$

6. Let the roots be $\pm\alpha$ and $\beta$. Since $\alpha + (-\alpha) + \beta = \frac{16}{4}$, we have $\beta = 4$. Since $\alpha(-\alpha)\beta = -\frac{36}{4}$, we have $\alpha = \pm\frac{3}{2}$.

8. Let the roots be $\frac{\alpha}{r}$, $\alpha$ and $\alpha r$. Then $(\frac{\alpha}{r})\alpha(\alpha r) = \frac{24}{3}$, so that $\alpha = 2$. Factoring out $x - 2$, we have $0 = (x-2)(3x^2 - 20x + 12) = (x-2)(3x-2)(x-6)$. Hence the roots are 2, $\frac{2}{3}$ and 6.

10. Let the roots be $\alpha$, $2\alpha$ and $\beta$. Then $\alpha + 2\alpha + \beta = -\frac{23}{12}$, $\alpha(2\alpha) + \alpha\beta + (2\alpha)\beta = \frac{3}{8}$ and $\alpha(2\alpha)\beta = \frac{3}{8}$. From the first equation, $\beta = -3\alpha - \frac{23}{12}$. Substituting into the second equation, we have $2\alpha^2 + 3\alpha(-3\alpha - \frac{23}{12}) = \frac{3}{8}$. This is equivalent to $0 = 56\alpha^2 + 46\alpha + 3 = (14\alpha + 1)(4\alpha + 3)$. If $\alpha = -\frac{1}{14}$,

then $\beta = \frac{3}{14} - \frac{23}{12} = -\frac{143}{84}$, but $2\alpha^2\beta \neq \frac{3}{8}$. If $\alpha = -\frac{3}{4}$, then $\beta = \frac{9}{4} - \frac{23}{12} = \frac{1}{3}$, and the third equation is satisfied this time. Hence the roots are $\frac{3}{4}$, $-\frac{3}{2}$ and $\frac{1}{3}$.

12. Let the roots be $\frac{\alpha}{r}$, $\alpha$ and $\alpha r$. Then $(\frac{\alpha}{r})\alpha(\alpha r) = -\frac{8}{27}$, so that $\alpha = -\frac{2}{3}$. Factoring out $3x+2$, we have $0 = (3x+2)(18x^2 - 25x + 8) = (3x+2)(9x-8)(2x-1)$. Hence the roots are $\frac{8}{9}$, $-\frac{2}{3}$ and $\frac{1}{2}$.

14. Let the roots be $\alpha$, $\beta$, $\gamma$ and $\delta$ with $\alpha\beta = 2$. Then $\gamma\delta = -1$. We have $(\alpha+\beta)+(\gamma+\delta) = \frac{29}{6}$ while $\frac{7}{6} = \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = 2(\gamma+\delta)-(\alpha+\beta)$. Addition yields $3(\gamma+\delta) = 6$ so that $\gamma+\delta = 2$, and subtraction yields $\alpha+\beta = \frac{17}{6}$. Now $\alpha$ and $\beta$ are the roots of $x^2 - \frac{17}{6}x + 2 = 0$. This is equivalent to $0 = 6x^2 - 17x + 12 = (2x-3)(3x-4)$. Hence they are $\frac{3}{2}$ and $\frac{4}{3}$. On the other hand, $\gamma$ and $\delta$ are the roots of $x^2 - 2x - 1 = 0$. Hence they are equal to $\frac{2\pm\sqrt{4+4}}{2} = 1 \pm \sqrt{2}$.

16. Let the roots be $\alpha < \beta < \gamma < \delta$ with $\alpha\delta = \beta\gamma$. Since $\alpha\beta\gamma\delta = \frac{64}{9}$, we have $\alpha\delta = \beta\gamma = \frac{8}{3}$. Now $(\alpha+\delta)+(\beta+\gamma) = \frac{65}{9}$ while $\frac{494}{27} = \alpha\delta+\beta\gamma+\alpha\beta+\alpha\gamma+\beta\delta+\gamma\delta = \frac{16}{3}+(\alpha+\delta)(\beta+\gamma)$ so that $(\alpha+\delta)(\beta+\gamma) = \frac{350}{27}$. It follows that $\alpha+\delta$ and $\beta+\gamma$ are the roots of $x^2 - \frac{65}{9}x + \frac{350}{27} = 0$. This is equivalent to $0 = 27x^2 - 65x + 350 = (9x-35)(3x-10)$. Since $\frac{10}{3} < \frac{35}{9}$, $\beta+\gamma = \frac{10}{3}$ while $\alpha+\delta = \frac{35}{9}$. Note that $\beta$ and $\gamma$ are the roots of $x^2 - \frac{10}{3}x + \frac{8}{3} = 0$. This is equivalent to $0 = 3x^2 - 10x + 8 = (3x-4)(x-2)$. Hence $\beta = \frac{4}{3}$ and $\gamma = 2$. On the other hand, $\alpha$ and $d$ are the roots of $x^2 - \frac{35}{9}x + \frac{8}{3} = 0$. This is equivalent to $0 = 9x^2 - 35x + 24 = (9x-8)(x-3)$. Hence $\alpha = \frac{9}{8}$ and $\delta = 3$.

18. (a) Note that $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma) = q^2 - 2pr$. Hence $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2} = \frac{q^2-2pr}{r^2}$.

    (b) Note that $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = p^2 - 2q$. It follows that $\frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2} + \frac{1}{\alpha^2\beta^2} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2\beta^2\gamma^2} = \frac{p^2-2q}{r^2}$.

20. Let the roots be $\alpha$, $\beta$, $\gamma$ and $\delta$. Note that $\alpha+\beta+\gamma+\delta = 0$. Now

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) = -2q.$$

Hence

$$\begin{aligned}
0 &= (\alpha+\beta+\gamma+\delta)^3 \\
&= (\alpha^3+\beta^3+\gamma^3+\delta^3) + 6(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta) \\
&\quad +3(\alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\beta^2\gamma+ \\
&\quad \beta^2\delta+\beta^2\alpha+\gamma^2\delta+\gamma^2\alpha+\gamma^2\beta+\delta^2\alpha+\delta^2\beta+\delta^2\gamma) \\
&= -2(\alpha^3+\beta^3+\gamma^3+\delta^3) + 6(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta) \\
&\quad +3(\alpha+\beta+\gamma+\delta)(\alpha^2+\beta^2+\gamma^2+\delta^2) \\
&= -2(\alpha^3+\beta^3+\gamma^3+\delta^3) + 6(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta).
\end{aligned}$$

Hence $\alpha^3+\beta^3+\gamma^3+\delta^3 = -3r$.

## Answers to Odd-numbered Exercises VI

1. $6x^4 - 13x^3 - 12x^2 + 39x - 18 = 0$.

3. $x^6 - 5x^5 - 8x^4 + 40x^3 + 16x^2 - 80x = 0$.

5. 1,3,5,7.

7. $\frac{1}{2}, \frac{1}{2}, -6$.

9. $-\frac{3}{2}, -2, 4$

11. $\pm\sqrt{3}, \frac{3}{4}, -\frac{1}{2}$.

13. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.

15. $-4, -1, 2, 5$.

17. $-\frac{4}{3}, -\frac{3}{2}, -\frac{5}{3}$.

19. (a) $-6q$; (b) $\frac{q}{r}$.

21. $2q^2$.

## Commentary VI

This chapter moves beyond the quadratic case to make statements about polynomial functions of general degree. The Remainder, Factor, and Fundamental theorems in addition to the extension of Vieta's theorem are introduced. The theorems are important and clarify the earlier remarks about roots and factoring. The examples and exercises focus on extracting roots with some relationship given. This situation is somewhat artificial, but leads to interesting problems and provides a good opportunity for the student to practice algebraic manipulation at the same time as applying a theorem.

Example 5 uses the notation of symmetric functions without which it becomes overly laborious. It was ommited in the lectures and moved to that chapter in Part 2.

The Fundamental Theorem is a wonderful result to know as part of the culture of this subject. Moreover, the result is a crucial motivator for the difficult sections which follow in preparation for the cubic and quartic. Since there always are roots, how do we go about finding them?

In Part 2 this chapter is expanded with the addition of the Rational Root Theorem, conjugate theorems, and transformations. In this way all of the work on equations of general degree is collected in the chapter.

## CHAPTER VII — THE ROOTS OF EQUATIONS

We begin with two important results.

**Conjugate Imaginary Roots Theorem.**
In a polynomial equation with real coefficients, imaginary roots occur in conjugate pairs.

**Proof:**

Suppose that $f(x) = 0$ is a polynomial equation with real coefficients, and suppose that it has an imaginary root $a + bi$, $b \neq 0$. We shall show that $a - bi$ is also a root.

The factor of $f(x)$ corresponding to these two roots is $(x - a - bi)(x - a + bi) = (x - a)^2 + b^2$. Let $f(x)$ be divided by $(x - a)^2 + b^2$. Denote the quotient by $Q(x)$, and the remainder, if any, by $Rx + R'$. Then $f(x) = Q(x)((x - a)^2 + b^2) + Rx + R'$.

In this identity put $x = a + bi$. Then $f(x) = 0$ by hypothesis. Also, $(x - a)^2 + b^2 = 0$. Hence $R(a + bi) + R' = 0$. Equating to zero the real and imaginary parts, $Ra + R' = 0$ and $Rb = 0$. Since $b \neq 0$ by hypothesis, we have $R = 0$ and $R' = 0$. Hence $f(x)$ is divisible by $(x - a)^2 + b^2$, that is, by $(x - a - bi)(x - a + bi)$. Hence $x = a - bi$ is also a root.

In the preceding result, we have seen that if the equation $f(x) = 0$ has a pair of conjugate imaginary roots $a \pm bi$, then $(x - a)^2 + b^2$ is a factor of the expression $f(x)$. Suppose that $a \pm bi$, $c \pm di$, $e \pm gi$, $\ldots$ are the imaginary roots of the equation $f(x) = 0$, and that $\Phi(X)$ is the product of the quadratic factors corresponding to these imaginary roots; then

$$\Phi(x) = ((x - a)^2 + b^2)((x - c)^2 + d^2)((x - e)^2 + g^2) + \cdots.$$

Now each of these factors is positive for every real value of $x$. Hence $\Phi(x)$ is always positive for real values of $x$.

Using the same technique as in the preceding proof, we can establish the following result.

**Conjugate Irrational Roots Theorem.**
In a polynomial equation with rational coefficients, irrational roots occur in conjugate pairs.

In other words, if $a + \sqrt{b}$ is a root, then $a - \sqrt{b}$ is also a root.

**Example 1.**
Solve the equation $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, having given that one root is $2 - \sqrt{3}$.

**Solution:**

Since $2 - \sqrt{3}$ is a root, we know that $2 + \sqrt{3}$ is also a root. Corresponding to this pair of roots, we have the quadratic factor $x^2 - 4x + 1$. Also, $6x^4 -$

$13x^3 - 35x^2 - x + 3 = (x^2 - 4x + 1)(6x^2 + 11x + 3)$. Hence the other roots are obtained from $0 = 6x^2 + 11x + 3 = (3x + 1)(2x + 3)$. Thus the roots are $-\frac{1}{3}$, $-\frac{3}{2}$ and $2 \pm \sqrt{3}$.

**Example 2.**

Form the equation of the fourth degree with rational coefficients, one of whose roots is $\sqrt{2} + \sqrt{3}i$.

**Solution:**

Here we must have $\sqrt{2} \pm \sqrt{3}i$ as one pair of roots, and $-\sqrt{2} \pm \sqrt{3}i$ as another pair. Corresponding to the first pair we have the quadratic factor $x^2 - 2\sqrt{2}x + 5$, and corresponding to the second pair we have the quadratic factor $x^2 + 2\sqrt{2}x + 5$. Thus the required equation is

$$(x^2 + 2\sqrt{2}x + 5)(x^2 - 2\sqrt{2}x + 5) = 0,$$

which may be rewritten as $(x^2 + 5)^2 - 8x^2 = 0$ or $x^4 + 2x^2 + 25 = 0$.

**Example 3.**

Show that the equation $\frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \cdots + \frac{H}{x-h} = k$ has no imaginary roots.

**Solution:**

If there is an imaginary root $p + qi$, then $p - qi$ is also a root. Substitute these values for $x$ and subtract the first result from the second. The result is

$$q\left(\frac{A^2}{(p-a)^2 + q^2} + \frac{B^2}{(p-b)^2 + q^2} + \frac{C^2}{(p-c)^2 + q^2} + \cdots + \frac{H^2}{(p-h)^2 + q^2}\right) = 0,$$

which is impossible unless $q = 0$.

To determine the nature of some of the roots of an equation, it is not always necessary to solve it. For instance, the truth of the following statements will be readily admitted.

- If the coefficients are all positive, the equation has no positive root. Thus the equation $x^4 + x^3 + 2x + 1 = 0$ cannot have a positive root.

- If the coefficients of the even powers of $x$ are all of one sign, and the coefficients of the odd powers are all of the contrary sign, the equation has no negative root. Thus the equation $x^7 + x^5 - 2x^4 + x^3 - 3x^2 + 7x - 5 = 0$ cannot have a negative root.

- If the equation contains only even powers of $x$ and the coefficients are all of the same sign, the equation has no real root. Thus the equation $2x^8 + 3x^4 + x^2 + 7 = 0$ cannot have a real root.

- If the equation contains only odd powers of $x$, and the coefficients re all of the same sign, the equation has no real root except $x = 0$. Thus the equation $x^9 + 2x^5 + 3x^3 + x = 0$ has no real root except $x = 0$.

All the foregoing results are included in the next result.

**Descartes' Rule of Signs.**
A polynomial equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$, and cannot have more negative roots than there are changes of sign in $f(-x)$.

**Proof:**

Suppose that the signs of the terms in a polynomial are

$$+ \quad + \quad - \quad - \quad + \quad - \quad - \quad - \quad + \quad - \quad + \quad -.$$

We shall show that if this polynomial is multiplied by a binomial whose signs are + -, there will be at least one more change of sign in the product than in the original polynomial. Writing down only the signs of the terms in the multiplication, we have:

$$\begin{array}{ccccccccccccc}
 & + & + & - & - & + & - & - & - & + & - & + & - \\
 & & & & & & & & & & & + & - \\
\hline
 & - & - & + & + & - & + & + & + & - & + & - & + \\
+ & + & - & - & + & - & - & - & + & - & + & - & \\
\hline
+ & \pm & - & \mp & + & - & \mp & \mp & + & - & + & - & +.
\end{array}$$

Hence ve see that in the product:

(i) an ambiguity replaces each continuation of sign in the original polynomial;

(ii) the signs before and after an ambiguity or set of ambiguities are unlike;

(iii) a change of sign is introduced at the end.

Let us take the most unfavorable case and suppose that all the ambiguities are replaced by continuations; from (ii) we see that the number of changes of sign will be the same whether we take the upper or the lower signs; let us take the upper; thus the number of changes of sign cannot be less than in

$$+ \quad \pm \quad - \quad \mp \quad + \quad - \quad \mp \quad \mp \quad + \quad - \quad + \quad - \quad +.$$

and this series of signs is the same as in the original polynomial with an additional chance of sign at the end. If we then suppose the factors corresponding to the negative and imaginary roots to be already multiplied together, each factor $x - a$ corresponding to a positive root introduces at least one change of sign. Therefore no equation can have more positive roots than it has changes of sign.

Again, the roots of the equation $f(-x) = 0$ are equal to those of $f(x) = 0$ but opposite to them in sign. Therefore the negative roots of $f(x) = 0$ are the positive roots of $f(-x) = 0$. The number of these positive roots cannot exceed the number of changes of sign in $f(-x)$, that is, the number of negative roots of $f(x) = 0$ cannot exceed the number of changes of sign in $f(-x)$.

**Example 4.**
Find the nature of the roots of the equation $x^9 + 5x^8 - x^3 + 7x + 2 = 0$.

**Solution:**

There are two changes of sign in $f(x)$. Therefore there are at most two positive roots. Now $f(-x) = -x^9 + 5x^8 + x^3 - 7x + 2$ has three changes of sign. Therefore the given equation has at most three negative roots. Hence it must have at least four imaginary roots.

## EXERCISES VII

Solve the equations:

1. $3x^4 - 10x^3 + 4x^2 - x - 6 = 0$, one root being $\frac{1+\sqrt{3}i}{2}$.

2. $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, one root being $2 - \sqrt{3}$.

3. $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, one root being $-1 + i$.

4. $x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$, one root being $i$.

5. $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $\sqrt{3}$ and another $1 - 2i$.

Form the equation of lowest degrees with rational coefficients, one of whose roots is:

6. $\sqrt{3} + \sqrt{2}i$. 7. $\sqrt{5} - i$. 8. $-\sqrt{2} - \sqrt{2}i$. 9. $\sqrt{5 + 2\sqrt{6}}$.

Form the equation whose roots are:

10. $\pm 4\sqrt{3}, 5 \pm 2i$. 11. $1 \pm \sqrt{2}i, 2 \pm \sqrt{3}i$.

12. Form the equation of the eighth degree with rational coefficients, one of whose roots is $\sqrt{2} + \sqrt{3} + i$.

13. Find the nature of the roots of the equation $3x^4 + 12x^2 + 5x - 4 = 0$.

14. Show that the equation $2x^7 - x^4 + 4x^3 - 5 = 0$ has at least four imaginary roots.

15. What may be inferred respecting the roots of the equation $x^{10} - 4x^6 + x^4 - 2x - 3 = 0$?

16. Find the least possible number of imaginary roots of the equation $x^9 - x^5 + x^4 + x^2 + 1 = 0$.

17. Find the condition that $x^3 - px^2 + qx - r = 0$ may have
    (a) two roots equal but of opposite sign;
    (b) the roots in geometrical progression.

18. Consider the roots of the equation $x^4 + px^3 + qx^2 + rx + s = 0$.
    (a) If they are in arithmetical progression, show that $p^3 - 4pq + 8r = 0$.
    (b) If they are in geometrical progression, show that $p^2 s = r^2$.

19. If the roots of the equation $x^n - 1 = 0$ are $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$, show that $(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1}) = n$.

If $\alpha$, $\beta$ and $\gamma$ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the value of:

20. $\sum \alpha^2\beta^2$. 21. $\sum \alpha + \beta$. 22. $\sum(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})$. 23. $\sum \alpha^2\beta$.

If $\alpha$, $\beta$, $\gamma$ and $\delta$ are the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, find the value of:

24. $\sum \alpha^2\beta\gamma$. 25. $\sum \alpha^4$.

## Solutions to Even-numbered Exercises VII

2. We must have $2+\sqrt{3}$ as another root. Now $(x-(2-\sqrt{3}))(x-(2+\sqrt{3})) = x^2 - 4x + 1$. Factoring out $x^2 - 4x + 1$, we have $0 = (x^2 - 4x + 1)(6x^2 + 11x + 3) = (x^2 - 4x + 1)(3x + 1)(2x + 3)$. Hence the other two roots are $-\frac{1}{3}$ and $-\frac{3}{2}$.

4. We must have $-i$ as another root. Now $(x+i)(x-i) = x^2 + 1$. Factoring out $x^2 + 1$, we have $0 = (x^2 + 1)(x^2 + 4x + 5)$. Hence the other two roots are $\frac{-4\pm\sqrt{16-20}}{2} = -2 \pm i$.

6. We must have

$$\begin{aligned}
0 &= (x - (\sqrt{3} + \sqrt{2}i))(x - (\sqrt{3} - \sqrt{2}i))(x - (-\sqrt{3} + \sqrt{2}i))(x - (-\sqrt{3} - \sqrt{2}i)) \\
&= ((x - \sqrt{3})^2 + 2)((x + \sqrt{3})^2 + 2) \\
&= (x^2 + 5 - 2\sqrt{3}x)(x^2 + 5 + 2\sqrt{3}x) \\
&= (x + 5)^2 - (2\sqrt{3}x)^2 \\
&= x^4 - 2x^2 + 25.
\end{aligned}$$

8. We must have

$$\begin{aligned}
0 &= (x-(\sqrt{2}+\sqrt{2}i))(x-(\sqrt{2}-\sqrt{2}i))(x-(-\sqrt{2}+\sqrt{2}i))(x-(-\sqrt{2}-\sqrt{2}i))\\
&= ((x-\sqrt{2})^2+2)((x+\sqrt{2})^2+2)\\
&= (x^2+4-2\sqrt{2}x)(x^2+4+2\sqrt{2}x)\\
&= (x^2+4)^2-(2\sqrt{2}x)^2\\
&= x^4+16.
\end{aligned}$$

10. We have

$$\begin{aligned}
0 &= (x-4\sqrt{3})(x+4\sqrt{3})(x-(5+2i))(x-(5-2i))\\
&= (x^2-48)((x-5)^2)+4)\\
&= x^4-10x^3-19x^2+480x-1392.
\end{aligned}$$

12. We must have

$$\begin{aligned}
0 &= (x-(\sqrt{2}+\sqrt{3}+i))(x-(\sqrt{2}+\sqrt{3}-i))(x-(\sqrt{2}-\sqrt{3}+i))\\
&\quad (x-(\sqrt{2}-\sqrt{3}-i))(x-(-\sqrt{2}+\sqrt{3}+i))(x-(-\sqrt{2}+\sqrt{3}-i))\\
&\quad (x-(-\sqrt{2}-\sqrt{3}+i))(x-(-\sqrt{2}-\sqrt{3}-i))\\
&= ((x-\sqrt{2}-\sqrt{3})^2+1)((x-\sqrt{2}+\sqrt{3})^2+1)\\
&\quad ((x+\sqrt{2}-\sqrt{3})^2+1)((x+\sqrt{2}+\sqrt{3})^2+1)\\
&= (x^2-2\sqrt{2}x-2\sqrt{3}x+6+2\sqrt{6})(x^2-2\sqrt{2}x+2\sqrt{3}x+6-2\sqrt{6})\\
&\quad (x^2+2\sqrt{2}x-2\sqrt{3}x+6-2\sqrt{6})(x^2+2\sqrt{2}x+2\sqrt{3}x+6+2\sqrt{6})\\
&= ((x^2-2\sqrt{2}x+6)^2-(2\sqrt{3}x-2\sqrt{6})^2)((x^2+2\sqrt{2}x+6)^2-(2\sqrt{3}x+2\sqrt{6})^2)\\
&= (x^4-4\sqrt{2}x^3+20x^2-24\sqrt{2}x+36-12x^2+24\sqrt{2}x-24)\\
&\quad (x^4+4\sqrt{2}x^3+20x^2+24\sqrt{2}x+36-12x^2-24\sqrt{2}x-24)\\
&= (x^4+8x^2+12-4\sqrt{2}x^3)(x^4+8x^2+12+4\sqrt{2}x^3)\\
&= (x^4+8x^2+12)^2-(4\sqrt{2}x^3)^2\\
&= x^8-16x^6+88x^4+192x^2+144.
\end{aligned}$$

14. Since there are 3 changes of signs, the number of positive roots is at most 3. Letting $x=-y$, we have $-2y^7-y^4-4y^3-5=0$. Since there are no changes of signs, there are no negative roots. It follows that there must be at least 4 imaginary roots.

16. Since there are 2 changes of signs, the number of positive roots is at most 2. Letting $x=-y$, we have $-y^9+y^5+y^4+y^2+1=0$. Since there is only 1 change of signs, the number of negative roots is at most 1. It follows that there must be at least 6 imaginary roots.

18. (a) Let the roots be $\alpha-3d,\ \alpha-d,\ \alpha+d$ and $\alpha+3d$. Then

$$\begin{aligned}
-p &= (\alpha-3d)+(\alpha-d)+(\alpha+d)+(\alpha+3d)\\
&= 4\alpha,\\
q &= (\alpha-3d)(\alpha+3d)+(\alpha-d)(\alpha+d)+(\alpha-3d)(\alpha-d)\\
&\quad +(\alpha+3d)(\alpha+d)+(\alpha-3d)(\alpha+d)+(\alpha+3d)(\alpha-d)\\
&= 6\alpha^2-10d^2,\\
-r &= (\alpha-3d)(\alpha+3d)(\alpha-d)+(\alpha-3d)(\alpha+3d)(\alpha+d)\\
&\quad +(\alpha-3d)(\alpha-d)(\alpha+d)+(\alpha+3d)(\alpha-d)(\alpha+d)\\
&= 4\alpha^3-20\alpha d^2.
\end{aligned}$$

Hence $p^3-4pq+8r = (-4\alpha)^3-4(-4\alpha)(6\alpha^2-10d^2)+8(-4\alpha^3+20\alpha d^2)=0$.

(b) Let the roots be $\frac{\alpha}{r^3}$, $\frac{\alpha}{r}$, $\alpha r$ and $\alpha r^3$. Then $-p=\frac{\alpha}{r^3}+\frac{\alpha}{r}+\alpha r+\alpha r^3$ while $s=(\frac{\alpha}{r^3})(\frac{\alpha}{r})(\alpha r)(\alpha r^3)=\alpha^4$. Hence

$$\begin{aligned}
-r &= (\frac{\alpha}{r^3})(\frac{\alpha}{r})(\alpha r)+(\frac{\alpha}{r^3})(\frac{\alpha}{r})(\alpha r^3)+(\frac{\alpha}{r^3})(\alpha r)(\alpha r^3)+(\frac{\alpha}{r})(\alpha r)(\alpha r^3)\\
&= \alpha^2(\frac{\alpha}{r^3}+\frac{\alpha}{r}+\alpha r+\alpha r^3)\\
&= -p\sqrt{s},
\end{aligned}$$

so that $r^2=p^2s$.

20. We have $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)=q^2-2pr$.

22. We have

$$\begin{aligned}
&\frac{\alpha}{\beta}+\frac{\beta}{\alpha}+\frac{\beta}{\gamma}+\frac{\gamma}{\beta}+\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma}\\
=\ &\frac{\alpha^2\gamma+\beta^2\gamma+\beta^2\alpha+\gamma^2\alpha+\gamma^2\beta+\alpha^2\beta}{\alpha\beta\gamma}\\
=\ &\frac{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma}-3\\
=\ &\frac{pq}{r}-3.
\end{aligned}$$

24. We have

$$\begin{aligned}
&\alpha^2\beta\gamma+\alpha^2\beta\delta+\alpha^2\gamma\delta+\beta^2\gamma\delta+\beta^2\gamma\alpha+\beta^2\delta\alpha\\
&+\gamma^2\delta\alpha+\gamma^2\delta\beta+\gamma^2\alpha\beta+\delta^2\alpha\beta+\delta^2\alpha\gamma+\delta^2\beta\gamma\\
=\ &(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta)-4\alpha\beta\gamma\delta\\
=\ &(-p)(-r)-4s\\
=\ &pr-4s.
\end{aligned}$$

## Answers to Odd-numbered Exercises VII

1. $3, -\frac{2}{3}, \frac{1\pm\sqrt{3}i}{2}$.

3. $-1 \pm \sqrt{2}, -1 \pm i$.

5. $-1, \pm\sqrt{3}, 1 \pm 2i$.

7. $x^4 - 8x^2 + 36 = 0$.

9. $x^4 - 10x^2 + 1 = 0$.

11. $x^4 - 6x^3 + 18x^2 - 26x + 21 = 0$.

13. One positive, one negative, two imaginary.

15. At least one positive, at least one negative, at least four imaginary.

17. (a) $pq = r$; (b) $p^3r = q^3$.

21. $pq - r$.

23. $pq - 3r$.

25. $p^4 - 4p^2q + 2q^2 + 4pr - 4s$.

## Commentary VII

This short chapter states two results on conjugate roots and describes Descartes' rule of signs. The significant corollary of the conjugate imaginary roots theorem is that every cubic polynomial real coefficients has at least one root. This is included as an example in Part 2.

The statement of the conjugate irrational root theorem was clarified and restricted to quadratic irrationals, that is square roots of non-square rationals.

We do not feel that Descartes' rule of signs is a very important topic, but include it for completeness. It may be easily ommited.

Exercises 20-25 belong with Example 5 of Chapter VI. Both the example and the exercises have been moved into the new chapter on Symmetric Identities.

# CHAPTER VIII — TRANSFORMATION OF EQUATIONS

The discussion of an equation is sometimes simplified by transforming it into another equation whose roots bear some assigned relation to those of the one proposed. Such transformations are especially useful in the solution of cubic equations.

Suppose we wish to transform an equation into another whose roots are those of the proposed equation with contrary signs. Let $f(x) = 0$ be the proposed equation. Put $-y$ for $x$. Then the equation $f(-y) = 0$ is satisfied by every root of $f(x) = 0$ with its sign changed. Thus the required equation is $f(-y) = 0$. If the proposed equation is $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$, then the required equation will be $p_0y^n - p_1y^{n-1} + p_2y^{n-2} - \cdots + (-1)^{n-1}p_{n-1}y + (-1)^n p_n = 0$, which is obtained from the original equation by changing the sign of every alternate term beginning with the second.

Suppose we wish to transform an equation into another whose roots are equal to those of the proposed equation multiplied by a given quantity. Let $f(x) = 0$ be the proposed equation, and let $q$ denote the given quantity. Put $y = qx$, so that $x = \frac{y}{q}$. Then the required equation is $f(\frac{y}{q}) = 0$. The chief use of this transformation is to clear an equation of fractional coefficients.

**Example 1.**
Remove fractional coefficients from the equation $2x^3 - \frac{3}{2}x^2 - \frac{1}{8}x + \frac{3}{16} = 0$.

**Solution:**

Put $x = \frac{y}{q}$ and multiply each term by $q^3$. Thus $2y^3 - \frac{3}{2}qy^2 - \frac{1}{8}q^2y + \frac{3}{16}q^3 = 0$. By putting $q = 4$, all the terms become integral, and on dividing by 2, we obtain $y^3 - 3y^2 - y + 6 = 0$.

Suppose we wish to find the equation whose roots are the squares of those of a proposed equation. Let $f(x) = 0$ be the given equation. Putting $y = x^2$, we have $x = \sqrt{y}$. Hence the required equation is $f(\sqrt{y}) = 0$.

**Example 2.**
Find the equation whose roots are the squares of those of the equation $x^3 + px^2 + qx + r = 0$.

**Solution:**

Putting $x = \sqrt{y}$ and transposing, we have $(y+q)\sqrt{y} = -(py+r)$. Squaring both sides, we have $(y^2 + 2qy + q^2)y = p^2y^2 + 2pry + r^2$ or $y^3 + (2q - p^2)y^2 + (q^2 - 2pr)y - r^2 = 0$.

**Remark:**
Compare the solution given in Chapter VI, Exercise 1.

## Reciprocal Equations

Suppose we wish to transform an equation into another whose roots are the reciprocals of the roots of the proposed equation. Let $f(x) = 0$ be the proposed equation. Put $y = \frac{1}{x}$ so that $x = \frac{1}{y}$. Then the required equation is $f(\frac{1}{y}) = 0$. One of the chief uses of this transformation is to obtain the values of expressions which involve symmetrical functions of negative powers of the roots.

**Example 3.**
If $\alpha$, $\beta$ and $\gamma$ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$.

**Solution:**

Write $\frac{1}{y}$ for $x$, multiply by $y^3$ and change all the signs. Then the resulting equation will be $ry^3 - qy^2 + py - 1 = 0$, with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. Hence $\sum \frac{1}{\alpha} = \frac{q}{r}$, $\sum \frac{1}{\alpha\beta} = \frac{p}{r}$ so that $\sum \frac{1}{\alpha^2} = \frac{q^2-2pr}{r^2}$.

**Example 4.**
If $\alpha$, $\beta$ and $\gamma$ are the roots of $x^3 + 2x^2 - 3x - 1 = 0$, find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$.

**Solution:**

Writing $\frac{1}{y}$ for x, the transformed equation is $y^3+3y^2-2y-1 = 0$. The given expression is equal to the value of $S_3$ in this equation. Here $S_1 = -3$, $S_2 = (-3)^2 - 2(-2) = 13$ and $S_3 + 3S_2 - 2S_1 - 3 = 0$. Hence $S_3 = -42$.

If an equation is unaltered by changing $x$ into $\frac{1}{x}$, it is called a *reciprocal* equation. If the given equation is $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-2}x^2 + p_{n-1}x + p_n = 0$, the equation obtained by writing $\frac{1}{x}$ for $x$ and clearing of fractions is $p_nx^n + p_{n-1}x^{n-1} + p_{n-2}x^{n-2} + \cdots + p_2x^2 + p_1x + 1 = 0$. If these two equations are the same, we must have $p_1 = \frac{p_{n-1}}{p_n}$, $p_2 = \frac{p_{n-2}}{p_n}$, $\ldots$, $p_{n-2} = \frac{p_2}{p_n}$, $p_{n-1} = \frac{p_1}{p_n}$ and $p_n = \frac{1}{p_n}$.

From the last result we have $p_n = \pm 1$, and thus we have two classes of reciprocal equations.

(i) If $p_n = 1$, then $p_1 = p_{n-1}$, $p_2 = p_{n-2}$, $p_3 = p_{n-3}$, $\ldots$; in other words, the coefficients of terms equidistant from the beginning and end are equal.

(ii) If $p_n = -1$, then $p_1 = -p_{n-1}$, $p_2 = -p_{n-2}$, $p_3 = -p_{n-3}$, .... Hence if the equation is of degree $2m$, then $p_m = -p_m$ or $p_m = 0$. In this case the coefficients of terms equidistant from the beginning and end are equal in magnitude and opposite in sign, and if the equation is of an even degree, the middle term is missing.

Suppose that $f(x) = 0$ is a reciprocal equation. If it is of the first class and of an odd degree, it has a root $-1$, so that $f(x)$ is divisible by $x + 1$. If $Q(x)$ is the quotient, then $Q(x) = 0$ is a reciprocal equation of the first class and of an even degree.

If $f(x) = 0$ is of the second class and of an odd degree, it has a root 1, so that $f(x)$ is divisible by $x - 1$. As before, $Q(x) = 0$ is a reciprocal equation of the first class and of an even degree. If $f(x) = 0$ is of the second class and of an even degree, it has a root 1 and a root $-1$, so that $f(x)$ is divisible by $x^2 - 1$. As before, $Q(x) = 0$ is a reciprocal equation of the first class and of an even degree.

Hence any reciprocal equation is of an even degree with its last term positive, or can be reduced to this form, which may therefore be considered as the standard form of reciprocal equations.

A reciprocal equation of the standard form can be reduced to an equation of half its degree. Let the equation be $p_0x^{2m} + p_1x^{2m-1} + p_2x + 2m - 2 + \cdots + p_mx^m + \cdots + p_2x^2 + p_1x + p_0 = 0$. Dividing by $x^m$ and rearranging the terms, we have $p_0(x^m + \frac{1}{x_m}) + p_1(x^{m-1} + \frac{1}{x^{m-1}}) + p_3(x^{m-2} + \frac{1}{x^{m-2}}) + \cdots + p_m = 0$. Now $x^{k+1} + \frac{1}{x^{k+1}} = (x^k + \frac{1}{x^k})(x + \frac{1}{x}) - (x^{k-1} + \frac{1}{x^{k-1}})$. Verify.

Letting $z = x + \frac{1}{x}$ and giving in succession to $k$ the values 1, 2, 3, …, we obtain $x^2 + \frac{1}{x^2} = z^2 - 2$, $x^3 + \frac{1}{x^3} = z(z^2 - 2) - z = z^3 - 3z$, $x^4 + \frac{1}{x^4} = z(z^3 - 3z) - (z^2 - 2) = z^4 - 4z^2 + 2$, and so on. Generally $x^m + \frac{1}{x^m}$ is of degree $m$ in $z$, and therefore the equation in $z$ is of degree $m$.

## Linear Shifts

Suppose we wish to transform an equation into another whose roots exceed those of the proposed equation by a given quantity. Let $f(x) = 0$ be the proposed equation, and let $h$ be the given quantity. Put $y = x + h$ so that $x = y - h$. Then the required equation is $f(y - h) = 0$. Similarly $f(y + h) = 0$ is an equation whose roots are less by $h$ than those of $f(x) = 0$.

**Example 5.**
Find the equation whose roots exceed by 2 the roots of the equation $4x^4 + 32x^3 + 83x^2 + 76x + 21 = 0$.

**Solution:**

The required equation will be obtained by substituting $y - 2$ for $x$ in the proposed equation. We have

$$\begin{aligned} & 4(y-2)^4 + 32(y-2)^3 + 83(y-2)^2 + 76(y-2) + 21 \\ = \; & 4(y^4 - 8y^3 + 24y^2 - 32y + 16) + 32(y^3 - 6y^2 + 12y - 8) \\ & +83(y^2 - 4y + 4) + 76(y - 2) + 21 \\ = \; & 4y^4 - 13y^2 + 9. \end{aligned}$$

**Remark:**
From $0 = 4y^4 - 13y^2 + 9 = (4y^2 - 9)(y^2 - 1) = (2y + 3)(2y - 3)(y + 1)(y - 1)$, the roots of this equation are $\pm\frac{3}{2}$ and $\pm 1$. Hence the root of the proposed equation are $-\frac{1}{2}$, $-\frac{7}{2}$, $-1$ and $-3$.

The chief use of this linear shift is to remove some assigned term from an equation. Let the given equation be $p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n = 0$. If $y = x - h$, we obtain the new equation $p_0(y + h)^n + p_1(y + h)^{n-1} + p_2(y + h)x^{n-2} + \cdots + p_n = 0$. When arranged in descending powers of $y$, it becomes $p_0y^n + (np_0h + p_1)y^{n-1} + (\frac{n(n-1)}{2}p_0h^2 + (n - 1)p_1h + p_2)y^{n-2} + \cdots$.

If the term to be removed is the second, we put $np_0h + p_1 = 0$ so that $h = \frac{p_1}{np_0}$. If the term to be removed is the third we put $\frac{n(n-1)}{2}p_0h^2 + (n - 1)p_1h + p_2 = 0$, and so obtain a quadratic to find $h$. Similarly, we may remove any other assigned term. Sometimes it will be more convenient to proceed in a different way.

**Example 6.**
Remove the second term from the equation $px^3 + qx^2 + rx + s = 0$.

**Solution:**

Let $\alpha$, $\beta$ and $\gamma$ be the roots, so that $\alpha+\beta+\gamma = -\frac{q}{p}$. If we increase each of the roots by $\frac{q}{3p}$, then in the transformed equation the sum of the roots will be equal to $-\frac{q}{p} + \frac{q}{p} = 0$. In other words, the coefficient of the second term will be zero. Hence the required transformation will be effected by substituting $y - \frac{q}{3p}$ for $x$ in the given equation. It is $py^3 + (r - \frac{q^2}{3p})y + (\frac{2q^3}{27p^2} - \frac{qr}{3p} + s) = 0$.

We have seen many examples of how, from the equation $f(x) = 0$, we may form an equation whose roots are connected with those of the given equation by some assigned relation. Let $y$ be a root of the required equation. Then the transformed equation can usually be obtained by expressing $x$ as a function of $y$ by means of the assigned relation and substituting this value of $x$ in $f(x) = 0$. Sometimes, it is necessary to eliminate $x$ between $f(x) = 0$ and the assigned relation.

**Example 7.**
If $\alpha$, $\beta$ and $\gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$, form the equation whose roots are $\alpha - \frac{1}{\beta\gamma}$, $\beta - \frac{1}{\gamma\alpha}$ and $\gamma - \frac{1}{\alpha\beta}$.

**Solution:**

When $x = \alpha$ in the given equation, $y = \alpha - \frac{1}{\beta\gamma}$ in the transformed equation. Note that we have $\alpha - \frac{1}{\beta\gamma} = \alpha - \frac{\alpha}{\alpha\beta\gamma} = \alpha + \frac{\alpha}{r}$. Therefore the transformed equation will be obtained by the substitution $y = x + \frac{x}{r}$ or $x = \frac{ry}{1+r}$. Thus the required equation is $r^2y^3 + pr(1 + r)y^2 + q(1 + r)^2y + (1 + r)^3 = 0$.

**Example 8.**
Form the equation whose roots are the squares of the differences of the roots of $x^3 + qx + r = 0$.

**Solution:**

Let $\alpha$, $\beta$ and $\gamma$ be the roots of the cubic. Then the roots of the required equation are $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$ and $(\alpha-\beta)^2$. Now

$$\begin{aligned}(\beta-\gamma)^2 &= \beta^2+\gamma^2-2\beta\gamma\\ &= \alpha^2+\beta^2+\gamma^2-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha}\\ &= (\alpha+\beta+\gamma)^2-2(\beta\gamma+\gamma\alpha+\alpha\beta)-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha}\\ &= -2q-\alpha^2+\frac{2r}{\alpha}.\end{aligned}$$

When $x=\alpha$ in the given equation, $y=(\beta-\gamma)^2$ in the transformed equation. It follows that $y=-2q-x^2+\frac{2r}{x}$. Thus we have to eliminate $x$ between the equations $x^3+qx+r=0$ and $x^3+(2q+y)x-2r=0$. By subtraction, $(q+y)x=3r$ or $x=\frac{3r}{q+y}$. Substituting and reducing, we obtain $y^3+6qy^2+9q^2y+(27r^2+4q^3)=0$.

**Remark:**
If $\alpha$, $\beta$ and $\gamma$ are real, $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$ and $(\alpha-\beta)^2$ are all non-negative. Therefore, $27r^2+4q^3$ is non-positive. Hence in order that the equation $x^3+qx+r=0$ may have all its roots real, $27r^2+4q^3$ must be non-positive. In other words, $\frac{r^2}{4}+\frac{q^3}{27}$ must be non-positive.

If $27r^2+4q^3=0$, the transformed equation has one root zero. Therefore the original equation has two equal roots. If $27r^2+4q^3$ is positive, the transformed equation has a negative root. Therefore the original equation must have two imaginary roots, since it is only such a pair of roots which can produce a negative root in the transformed equation.

## EXERCISES VIII

1. Transform the equation $x^3-4x^2+\frac{1}{4}x-\frac{1}{5}=0$ into another with integral coefficients, and unity for the coefficient of the first term.
2. Transform the equation $3x^4-5x^3+x^2-x+1=0$ into another the coefficient of whose first term is unity.

Solve the equations:

3. $2x^4+x^3-6x^2+x+2=0$.

4. $x^4-10x^3+26x^2-10x+1=0$.

5. $x^5-5x^4+9x^3-9x^2+5x-1=0$.

6. $4x^6-24x^5+57x^4-73x^3+57x^2-24x+4=0$.

7. Solve the equation $3x^3-22x^2+48x-32=0$ the roots of which are in harmonical progression.

8. The roots of $x^3 - 11x^2 + 36x - 36 = 0$ are in harmonical progression. Find them.

9. If the roots of the equation $x^3 - ax^2 + x - b = 0$ are in harmonical progression, show that the mean root is $3b$.

10. Solve the equation $40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0$, the roots of which are in harmonical progression.

Remove the second term from the equations:

11. $x^3 - 6x^2 + 10x - 3 = 0$.

12. $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$.

13. $x^5 + 5x^4 + 3x^3 + x^2 + x - 1 = 0$.

14. $x^6 - 12x^5 + 3x^2 - 17x + 300 = 0$.

15. Transform the equation $x^3 - \frac{1}{4}x - \frac{3}{4} = 0$ into one whose roots exceed by 3 the corresponding roots of the given equation.

16. Diminish by 3 the roots of the equation $x^5 - 4x^4 + 3x^2 - 4x + 6 = 0$.

17. Find the equation each of whose roots is greater by 1 than the corresponding root of the equation $x^3 - 5x^2 + 6x - 3 = 0$.

18. Find the equation whose roots are the squares of the roots of $x^4 + x^3 + 2x^2 + x + 1 = 0$.

19. Form the equation whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.

If $\alpha$, $\beta$ and $\gamma$ are the roots of $x^3 + qx + r = 0$, form the equation whose roots are:

20. $k\alpha - 1, k\beta - 1, k\gamma - 1$.

21. $\beta^2\gamma^2, \gamma^2\alpha^2, \alpha^2\beta^2$.

22. $\frac{\beta+\gamma}{\alpha^2}, \frac{\gamma+\alpha}{\beta^2}, \frac{\alpha+\beta}{\gamma^2}$.

23. $\beta\gamma + \frac{1}{\alpha}, \gamma\alpha + \frac{1}{\beta}, \alpha\beta + \frac{1}{\gamma}$.

24. $\alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta)$.

25. $\alpha^3, \beta^3, \gamma^3$.

26. $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

27. Show that the cubes of the roots of $x^3 + ax^2 + bx + ab = 0$ are given by the equation $x^3 + a^3x^2 + b^3x + a^3b^3 = 0$.

28. Solve the equation $x^5 - 5x^4 - 5x^3 + 25x^2 + 4x - 20 = 0$, whose roots are of the form $\alpha, -\alpha, \beta, -\beta, \gamma$.

29. If the roots of $x^3 + 3px^2 + 3qx + r = 0$ are in harmonical progression, show that $2q^3 = r(3pq - r)$.

## Solutions to Even-numbered Exercises VIII

2. We have $x^4 - \frac{5}{3}x^3 + \frac{1}{3}x^2 - \frac{1}{3}x + \frac{1}{3} = 0$. Let $x = \frac{1}{3}y$. Then $\frac{1}{81}y^4 - \frac{5}{81}y^3 + \frac{1}{27}y^2 - \frac{1}{9}y + \frac{1}{3} = 0$ or $y^4 - 5y^3 + 3y^2 - 9y + 27 = 0$.

4. The reciprocal equation may be rewritten as $(x^2+\frac{1}{x^2})-10(x+\frac{1}{x})+26 = 0$. Let $y = x + \frac{1}{x}$. Then $y^2 = x^2 + \frac{1}{x^2} + 2$. Hence $0 = y^2 - 10y + 24 = (y-4)(y-6)$ so that $y = 4$ or $6$. From $x+\frac{1}{x} = 4$, we have $x^2-4x+1 = 0$ so that $x = \frac{4\pm\sqrt{16-4}}{2} = 2 \pm \sqrt{3}$. From $x + \frac{1}{x} = 6$, we have $x^2 - 6x + 1 = 0$ so that $x = \frac{6\pm\sqrt{36-4}}{2} = 3 \pm 2\sqrt{2}$.

6. The reciprocal equation may be rewritten as $4(x^3 + \frac{1}{x^3}) - 24(x^2 + \frac{1}{x^2}) + 57(x + \frac{1}{x}) - 73 = 0$. Let $y = x + \frac{1}{x}$. Then $y^2 = x^2 + \frac{1}{x^2} + 2$ and $y^3 = x^3 + \frac{1}{x^3} + 3(x + \frac{1}{x})$. Hence $4y^3 - 24y^2 + 45y - 25 = 0$. By inspection, $y = 1$ is a root. From $x + \frac{1}{x} = 1$, we have $x^2 - x + 1 = 0$ so that $x = \frac{1\pm\sqrt{1-4}}{2} = \frac{1\pm\sqrt{3}i}{2}$. Factoring out $y - 1$, we have $0 = (y - 1)(4y^2 - 20y + 25) = (y - 1)(2y - 5)^2$. Hence we have a double root $y = \frac{5}{2}$. From $x + \frac{1}{x} = \frac{5}{2}$, we have $0 = 2x^2 - 5x + 2 = (2x - 1)(x - 2)$ so that $x = \frac{1}{2}$ or 2. Both of them are double roots.

8. Let the roots be $\alpha$, $\beta$ and their harmonic mean $\frac{2\alpha\beta}{a+b}$. From $36 = \alpha\beta + \frac{2\alpha^2\beta}{\alpha+\beta} + \frac{2\alpha\beta^2}{\alpha+\beta} = 3\alpha\beta$, we have $\alpha\beta = 12$. From $\alpha\beta(\frac{2\alpha\beta}{\alpha+\beta}) = 36$, we have $\alpha+\beta = 8$. Hence $\alpha$ and $\beta$ are the roots of the equation $0 = x^2-8x+12 = (x - 2)(x - 6)$. It follows that $\alpha = 2$, $\beta = 6$ and $\frac{2\alpha\beta}{\alpha+\beta} = 3$.

10. Let the middle two roots be $\alpha$ and $\beta$. Then the first and the last roots are $\frac{\alpha\beta}{2\alpha-\beta}$ and $\frac{\alpha\beta}{2\beta-\alpha}$. We have $\frac{1}{40} = (\frac{\alpha\beta}{2\beta-\alpha})\alpha\beta(\frac{\alpha\beta}{2\alpha-\beta}) = \frac{\alpha^3\beta^3}{(2\beta-\alpha)(2\alpha-\beta)}$ and $-\frac{1}{20} = \frac{\alpha^2\beta^2}{2\beta-\alpha} + \frac{\alpha^2\beta^2}{2\alpha-\beta} + \frac{\alpha^3\beta^2}{(2\beta-\alpha)(2\alpha-\beta)} + \frac{\alpha^2\beta^3}{(2\beta-\alpha)(2\alpha-\beta)} = \frac{2\alpha^2\beta^2(\alpha+\beta)}{(2\beta-\alpha)(2\alpha-\beta)}$. Dividing the first equation by the second, we have $\frac{\alpha\beta}{\alpha+\beta} = -1$. Let $k = \alpha\beta$. Then $\alpha + \beta = -k$ and $(2\beta - \alpha)(2\alpha - \beta) = 9\alpha\beta - 2(\alpha + \beta)^2 = 9k - 2k^2$. From $\frac{1}{40} = \frac{k^3}{9k-2k^2} = \frac{k^2}{9-2k}$, we have $0 = 40k^2 + 2k - 9 = (20k - 9)(2k + 1)$. From $\alpha\beta = k = -\frac{1}{2}$, we have $\alpha + \beta = \frac{1}{2}$ so that $\alpha$ and $\beta$ are the roots of $x^2 - \frac{1}{2}x - \frac{1}{2} = 0$. Hence $0 = 2x^2-x-1 = (2x+1)(x-1)$ so that $\alpha = 1$ and $\beta = -\frac{1}{2}$. It follows that $\frac{\alpha\beta}{2\beta-\alpha} = \frac{1}{4}$ and $\frac{\alpha\beta}{2\alpha-\beta} = -\frac{1}{5}$. From $\alpha\beta = k = \frac{9}{20}$, we have $\alpha + \beta = -\frac{9}{20}$ so that $\alpha$ and $\beta$ are the roots of $x^2 + \frac{9}{20}x + \frac{9}{20} = 0$. However, $20x^2 + 9x + 9$ is not a factor of $40x^4 - 22x^3 - 21x^2 + 2x + 1$ and $k = \frac{9}{20}$ must be rejected.

12. Let $x = y - 1$. Then

$$0 = (y - 1)^4 + 4(y - 1)^3 + 2(y - 1)^2 - 4(y - 1) - 2$$

$$\begin{aligned} &= y^4 - 4y^3 + 6y^2 - 4y + 1 + 4y^3 - 12y^2 + 12y - 4 \\ &\quad +2y^2 - 4y + 2 - 4y + 4 - 2 \\ &= y^4 - 4y^2 + 1. \end{aligned}$$

14. Let $x = y + 2$. Then

$$\begin{aligned} 0 &= (y+2)^6 - 12(y+2)^5 + 3(y+2)^2 - 17(y+2) + 300 \\ &= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64 \\ &\quad -12y^5 - 120y^4 - 480y^3 - 960y^2 - 960y - 384 + 3y^2 \\ &\quad +12y + 12 - 17y - 34 + 300 \\ &= y^6 - 60y^4 - 320y^3 - 717y^2 - 773y - 42. \end{aligned}$$

16. Let $y = x - 3$ so that $x = y + 3$. We have

$$\begin{aligned} 0 &= (y+3)^5 - 4(y+3)^4 + 3(y+3)^2 - 4(y+3) + 6 \\ &= y^5 + 15y^4 + 90y^3 + 270y^2 + 405y + 243 - 4y^4 - 48y^3 - 216y^2 \\ &\quad -432y - 324 + 3y^2 + 18y + 27 - 4y - 12 + 6 \\ &= y^5 + 11y^4 + 42y^3 + 57y^2 - 13y - 60. \end{aligned}$$

18. Let $y = x^2$ so that $x = \sqrt{y}$. We have $(\sqrt{y})^4 + (\sqrt{y})^3 + 2(\sqrt{y})^2 + \sqrt{y} + 1 = 0$. This is equivalent to $(y+1)^2 = -\sqrt{y}(y+1)$. Squaring yields $(y+1)^4 = y(y+1)^2$ or

$$0 = (y^2 + 2y + 1)(y^2 + y + 1) = y^4 + 3y^3 + 4y^2 + 3y + 1.$$

20. Let $y = \frac{k}{x}$ so that $x = \frac{k}{y}$. Then the desired equation is $(\frac{k}{y})^3 + q(\frac{k}{y}) + r = 0$ or $ry^3 + kqy^2 + k^3 = 0$.

22. Let $k = -1$ in Problem 20. Then the desired equation is $ry^3 - qy^2 - 1 = 0$.

24. Let $y = -x^2$ so that $x = \sqrt{y}i$. Then the desired equation is $(\sqrt{y}i)^3 + q(\sqrt{y}i) + r = 0$ or $r = \sqrt{y}i(y - q)$. Squaring yields $r^2 = -y(y^2 - 2qy + q^2)$ so that $y^3 - 2qy^2 + q^2y + r^2 = 0$.

26. First, note that $\frac{\beta}{\gamma} + \frac{\gamma}{\beta} = \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2+2\beta\gamma}{\beta\gamma} - 2 = \frac{\alpha(\beta+\gamma)^2}{\alpha\beta\gamma} - 2 = \frac{\alpha^3}{-r^2} - 2 = \frac{-\alpha^3-2r}{r}$. Let $y = \frac{-x^3-2r}{r}$ so that $x = -\sqrt[3]{r(y+2)}$. Then the desired equation is $-r(y+2) - q\sqrt[3]{r(y+2)} + r = 0$ or $-r(y+1) = q\sqrt[3]{r(y+2)}$. Cubing yields $-r^3(y^3 + 3y^2 + 3y + 1) = q^3r(y+2)$. This is equivalent to $r^3y^3 + 3r^3y^2 + (3r^2 + q^3)y + r(r^2 + 2q^3) = 0$.

28. We have $5 = \alpha + (-\alpha) + \beta + (-\beta) + \gamma = \gamma$ and $20 = (-\alpha^2)(-\beta^2)\gamma = 5\alpha^2\beta^2$ so that $\alpha\beta = 2$. Now $-5 = \alpha(-\alpha) + \beta(-\beta) + \alpha\beta + \alpha(-\beta) + (-\alpha)\beta + (-\alpha)(-\beta) + \alpha\gamma + (-\alpha)\gamma + \beta\gamma + (-\beta)\gamma = -\alpha^2 - \beta^2$. Hence $(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = 5 + 4 = 9$ so that $\alpha + \beta = 3$. Along with $\alpha\beta = 2$, we see that $\alpha$ and $\beta$ are the roots of $0 = x^2 - 3x + 2 = (x-1)(x-2)$. Hence $\alpha = 1$ and $\beta = 2$. The five roots are $\pm 1$, $\pm 2$ and 5.

## Answers to Odd-numbered Exercises VIII

1. $y^3 - 24y^2 + 9y - 24 = 0.$

3. $1, 1, -2, -\frac{1}{2}.$

5. $1, \frac{1 \pm \sqrt{3}i}{2}, \frac{3 \pm \sqrt{5}}{2}.$

7. $4, 2, \frac{4}{3}.$

11. $y^3 - 2y + 1 = 0.$

13. $y^5 - 7y^3 + 12y^2 - 7y = 0.$

15. $y^3 - \frac{9}{2}y^2 + \frac{13}{2}y - \frac{15}{4} = 0.$

17. $y^3 - 8y^2 + 19y - 15 = 0.$

19. $y^3 + 33y^2 + 12y + 8 = 0$

21. $y^3 - q^2y^2 - 2qr^2y - r^4 = 0.$

23. $ry^3 + q(1 - r)y^2 + (1 - r)^3 = 0.$

25. $y^3 + 3ry^2 + (q^3 + 3r^2)y + r^3 = 0.$

## Commentary VIII

This is a difficult chapter, however the payoff is that it provides an alternative perspective on some earlier problems and, most importantly, provides the mechanism for removing the second term of a cubic or quartic. The students managed to complete the exercises by carefully studying the examples and solved exercises. In Part 2 this chapter is given less emphasis by including it as a section alongside the other theory on higher order equations. Having covered the topic here, we permit ourselves later to assume that the second term may be eliminated without loss of generality.

# CHAPTER IX — CUBIC AND QUARTIC EQUATIONS

## Cubic Equations

The general type of a cubic equation is $x^3 + Px^2 + Qx + R = 0$, but as explained in Chapter VIII, this equation can be reduced to the simpler form $x^3 + qx + r = 0$, which we shall take as the standard form of a cubic equation.

To solve the equation $x^3 + qx + r = 0$. Let $x = y + z$. Then $x^3 = y^3 + z^3 + 3yz(y + z) = y^3 + z^3 + 3yzx$, and the given equation becomes $y^3 + z^3 + (3yz + q)x + r = 0$.

At present $y$ and $z$ are any two quantities subject to the condition that their sum is equal to one of the roots of the given equation. If we further suppose that they satisfy the equation $3yz + q = 0$, they are completely determinate. We thus obtain $y^3 + z^3 = -r$ and $y^3z^3 = -\frac{q^3}{27}$. Hence $y^3$ and $z^3$ are the roots of the quadratic $t^2 + rt - \frac{q^3}{27} = 0$. Solving this equation, and putting $y^3 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ and $z^3 = -\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$, we obtain the value of $x$ from the relation $x = y + z$. Thus

$$x = \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}}.$$

The above solution is generally known as Cardan's Solution, as it was first published by him in the Ars Magna, in 1545. Cardan obtained the solution from Tartaglia; but the solution of the cubic seems to have been due originally to Scipio Ferreo, about 1505.

By Chapter I, each of $y^3$ and $z^3$ has three cube roots. Hence it would appear that $x$ has *nine* values. This, however is not the case. Since $yz = -\frac{q}{3}$, the cube roots are to be taken in pairs so that the product of each pair is rational.

Hence if $y$ and $z$ denote the values of any pair of cube roots which fulfill this condition, the only other admissible pairs will be $\omega y$ and $\omega^2 z$, as well as $\omega^2 y$ and $\omega z$, where $\omega$ and $\omega^2$ are the imaginary cube roots of unity. Hence the roots of the equation are $y + z$, $\omega y + \omega^2 z$ and $\omega^2 y + \omega z$.

**Example 1.**
Solve the equation $x^3 - 15x = 126$.

**Solution:**

Let $x = y + z$. Then $y^3 + z^3 + (3yz - 15)x = 126$. Let $3yz - 15 = 0$. Then $y^3 + z^3 = 126$ and $y^3z^3 = 125$. Hence $y^3$ and $z^3$ are the roots of the equation $0 = t^2 - 126t + 125 = (t - 125)(t - 1)$. It follows that $y^3 = 125$ and $z^3 = 1$, so that $y = 5$ and $z = 1$. Thus the roots are $y + z = 5 + 1 = 6$, $\omega y + \omega^2 z = \frac{-5+5\sqrt{3}i}{2} + \frac{-1-\sqrt{3}i}{2} = -3 + 2\sqrt{3}i$ and $\omega^2 y + \omega z = -3 - 2\sqrt{3}i$.

To explain the reason why we apparently obtain nine values for $x$ earlier, we observe that $y$ and $z$ are to be found from the equations $y^3 + z^3 = -r$ and $yz = -\frac{q}{3}$. However, in the process of solution, the second of these was changed into $y^3z^3 = -\frac{q^3}{27}$, which would also hold if $yz = -\frac{\omega q}{3}$ or $yz = -\frac{\omega^2 q}{3}$. Hence the other six values of $x$ are solutions of the cubics $x^3 + \omega qx + r = 0$ and $x^3 + \omega^2 qx + r = 0$.

We proceed to consider more fully the roots of the equation $x^3 + qx + r = 0$.

(i) If $\frac{r^2}{4} + \frac{q^3}{27}$ is positive, then $y^3$ and $z^3$ are both real. Let $y$ and $z$ represent their arithmetical cube roots. Then the roots are $y + z$, $\omega y + \omega^2 z$ and $\omega^2 y + \omega z$. The first of these is real, and by substituting for $\omega$ and $\omega^2$ the other two become $-\frac{y+z}{2} + \frac{(y-z)\sqrt{3}i}{2}$ and $-\frac{y+z}{2} - \frac{(y-z)\sqrt{3}i}{2}$.

(ii) If $\frac{r^2}{4} + \frac{q^3}{27}$ is zero, then $y^3 = z^3$. In this case $y = z$, and the roots become $2y$, $y(\omega + \omega^2) = -y$ and $y(\omega^2 + \omega) = -y$.

(iii) If $\frac{r^2}{4} + \frac{q^3}{27}$ is negative, then $y^3$ and $z^3$ are imaginary expressions of the form $a + bi$ and $a - bi$. Suppose that the cube roots of these quantities are $m + ni$ and $m - ni$. Then the roots of the cubic become $(m + ni) + (m - ni) = 2m$, $(m + ni)\omega + (m - ni)\omega^2 = -m - n\sqrt{3}$ and $(m + ni)\omega^2 + (m - ni)\omega = -m + n\sqrt{3}$, which are all real quantities. As however there is no general arithmetical or algebraical method of finding the exact value of the cube root of imaginary quantities, the solution obtained earlier is of little practical use when the roots of the cubic are all real and unequal.

This case is sometimes called the *Irreducible Case* of Cardan's solution. It may be completed by the use of Trigonometry.

## Quartic Equations

We shall now give a brief discussion of some of the methods which are employed to obtain the general solution of a quartic equation. It will be found that in each of the methods we have first to solve an auxiliary cubic equation; and thus it will be seen that as in the case of the cubic, the general solution is not adapted for writing down the solution of a given numerical equation.

The solution of a quartic equation was first obtained by Ferrari, a pupil of Cardan, as follows. Denote the equation by $x^4 + 2px^3 + qx^2 + 2rx + s = 0$. Add to each side $(ax + b)^2$, the quantities $a$ and $b$ being determined so as to make the left side a perfect square. Then $x^4 + 2px^2 + (q + a^2)x^2 + 2(r + ab)x + (s + b^2) = (ax + b)^2$. Suppose that the left side of the equation is equal to $(x^2 + px + k)^2$. Then by comparing the coefficients, we have $p^2 + 2k = q + a^2$, $pk = r + ab$ and $k^2 = s + b^2$. By eliminating $a$ and $b$ from these equations, we obtain $(pk - r)^2 = (2k + p^2 - q)(k^2 - s)$ or $2k^3 - qk^2 + 2(pr - s)k - (p^2s - qs + r^2) = 0$.

From this cubic equation one real value of $k$ can always be found. Thus $a$ and $b$ are known. Also, $(x^2+px+k)^2 = (ax+b)^2$. It follows that $x^2+px+k = \pm(ax+b)$, and the values of $x$ are to be obtained from the two quadratics $x^2+(y-a)x+(k-b)=0$ and $x^2+(p+a)x+(k+b)=0$.

**Example 2.**
Solve the equation $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$.

**Solution:**

Add $a^2x^2 + 2abx + b^2$ to each side of the equation, and assume

$$x^4 - 2x^3 + (a^2-5)x^2 + 2(ab+5)x + (b^2-3) = (x^2 - x + k)^2.$$

Then by equating coefficients, we have $a^2 = 2k+6$, $ab = -k-5$ and $b^2 = k^2+3$. It follows that $(2k+6)(k^2+3) = (k+5)^2$ or $2k^3+5k^2-4k-7=0$. By trial, we find that $k=-1$. Hence $a^2=4$, $b^2=4$ and $ab=-4$. From the assumption, it follows that $(x^2-x+k)^2 = (ax+b)^2$. Substituting the values of $k$, $a$ and $b$, we have the two equations $x^2-x-1 = \pm(2x-2)$, that is, $x^2-3x+1=0$ and $x^2+x-3=0$. Thus the roots are $\frac{3\pm\sqrt{5}}{2}$ and $\frac{-1\pm\sqrt{13}}{2}$.

The following solution was given by Descartes in 1637. Suppose that the quartic equation is reduced to the form $x^4+qx^2+rx+s=0$. Assume that it can be factored as $(x^2+kx+m)(x^2-kx+n)$. Then by equating coefficients, we have $m+n-k^2=q$, $k(n-m)=r$ and $mn=s$. From the first two of these equations, we obtain $2n = k^2+q+\frac{r}{k}$ and $m = k^2+q-\frac{r}{k}$. Hence substituting in the third equation, $(k^3+qk+r)(k^3+qk-r)=4sk^2$ or $k^6+2qk^4+(q^2-4s)k^2-r^2=0$.

This is a cubic in $k^2$ which always has one real positive solution. Thus when $k^2$ is known the values of $m$ and $n$ are determined, and the solution of the quartic is obtained by solving the two quadratics $x^2+kx+m=0$ and $x^{-}kx+n=0$.

**Example 3.**
Solve the equation $x^4 - 2x^2 + 8x - 3 = 0$.

**Solution:**

Assume that $x^4-2x^2+8x-3 = (x^2+kx+m)(x^2-kx+n)$. Then by equating coefficients, we have $m+n-k^2=-2$, $k(n-m)=8$ and $mn=-3$. Thus we obtain $(k^3-2k+8)(k^3-2k-8)=-12k^2$, or $k^6-4k^4+16k^2-64=0$. This equation is clearly satisfied when $k^2-4=0$, or $k=\pm2$. It will be sufficient to consider one of the values of $k$. Putting $k=2$, we have $m+n=2$ and $n-m=4$, that is, $m=-1$ and $n=3$. Thus $x^4-2x^2+Sx-3 = (x^2+2x-1)(x^2-2x+3)$. Hence $x^2+2x-1=0$ or $x^2-2x+3=0$, and therefore the roots are $-1\pm\sqrt{2}$ and $1\pm\sqrt{2}i$.

There is no general algebraical solution of equations of a degree higher than the fourth. If, however, the coefficients of an equation are numerical, the value of any real root may be approximated to any required degree of accuracy.

## EXERCISES IX

Solve the following equations:

1. $x^3 - 18x = 35.$

2. $x^3 + 72x - 1720 = 0.$

3. $x^3 + 63x - 316 = 0.$

4. $x^3 + 21x + 342 = 0.$

5. $28x^3 - 9x^2 + 1 = 0.$

6. $x^3 - 15x^2 - 33x + 847 = 0.$

7. $2x^3 + 3x^2 + 3x + 1 = 0.$

8. Prove that the real root of the equation $x^3 + 12x - 12 = 0$ is $2(\sqrt[3]{2}) - \sqrt[3]{4}$.

Solve the following equations:

9. $x^4 - 3x^3 - 42x - 40 = 0.$

10. $x^4 - 10x^2 - 20x - 16 = 0.$

11. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0.$

12. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$

13. $x^4 - 3x^2 - 6x - 2 = 0.$

14. $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0.$

15. $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0.$

16. $x^5 - 6x^4 - 17x^3 + 17x^2 + 6x - 1 = 0.$

17. Solve $x^4 + 9x^3 + 12x^2 - 80x - 192 = 0$, which has equal roots.

18. (a) Find the relation between $q$ and $r$ in order that the equation $x^3 + qx + r = 0$ may be put into the form $x^4 = (x^2 + ax + b)^2$.
    (b) Solve the equation $8x^3 - 36x + 27 = 0$.

19. Consider the equations $x^{+}3px^2 + 3qx + r$ and $x^2 + 2px + q$.
    (a) If they have a common factor, show that $4(p^2 - q)(q^2 - pr) - (pq - r)^2 = 0$.
    (b) If they have two common factors, show that $p^2 - q = 0$ and $q^2 - pr = 0$.

20. If the equation $ax^3 + 3bx^2 + 3cx + d = 0$ has two equal roots, show that each of them is equal to $\frac{bc-ad}{2(ac-b^2)}$.

21. Show that the equation $x^4 + px^3 + qx^2 + rx + s = 0$ may be solved as a quadratic if $r^2 = p^2s$.

22. Solve the equation $x^6 - 18x^4 + 16x^3 + 28x^2 - 32x + 8 = 0$, one of whose roots is $\sqrt{6} - 2$.

23. If $\alpha$, $\beta$, $\gamma$ and $\delta$ are the roots of the equation $x^4 + qx^2 + rx + s = 0$, find the equation whose roots are $\beta + \gamma + \delta + (\beta\gamma\delta)^{-1}$, $\gamma + \delta + \alpha(\gamma\delta\alpha)^{-1}$, $\delta + \alpha + \beta + (\delta\alpha\beta)^{-1}$ and $\alpha + \beta + \gamma + (\alpha\beta\gamma)^{-1}$.

24. Consider the equation $x^4 - px^3 + qx^2 - rx + s = 0$.
    (a) Prove that $p^3 - 4pq + 8r = 0$ if the sum of two of the roots is equal to the sum of the other two roots.
    (b) Prove that $r^2 = p^2 s$ if the product of two of the roots is equal to the product of the other two roots.

25. Find the two roots of $x^5 - 209x + 56 = 0$ whose product is 1.

26. Find the two roots of $x^5 - 409x + 285 = 0$ whose sum is 5.

27. If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$, show that $(1 + \alpha_1^2)(1 + \alpha_2^2)\cdots(1 + \alpha_n^2) = (1 - p_2 + p_4 - \cdots)^2 + (p_1 - p_3 + p_5 - \cdots)^2$.

28. The sum of two roots of the equation $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ is 4. Solve the equation from the knowledge of this fact.

### Solutions to Even-numbered Exercises IX

2. We have $y^3 + z^3 = 1720$ while $y^3z^3 = -\frac{72^3}{27} = -13824$. From

$$0 = t^2 - 1720t - 13824 = (t - 1728)(t + 8),$$

we have $y = \sqrt[3]{1728} = 12$ and $z = \sqrt[3]{-8} = -2$. Hence the roots of the cubic equation are $y + z = 10$, $y\omega + z\omega^2 = -6 + 6\sqrt{3}i + 1 + \sqrt{3}i = -5 + 7\sqrt{3}i$ and $y\omega^2 + z\omega = -5 - 7\sqrt{3}i$.

4. We have $y^3 + z^3 = -342$ while $y^3z^3 = -\frac{21^3}{27} = -343$. From $0 = t^2 + 342t - 343 = (t - 1)(t + 343)$, we have $y\sqrt[3]{1} = 1$ and $z = \sqrt[3]{-343} = -7$. Hence the roots of the cubic equation are $y + z = -6$, $y\omega + z\omega^2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i + \frac{7}{2} + \frac{7\sqrt{3}}{2}i = 3 + 4\sqrt{3}i$ and $y\omega^2 + z\omega = 3 - 4\sqrt{3}i$.

6. Let $w = x - 5$ so that $x = w + 5$. Then

$$0 = (w + 5)^3 - 15(w + 5)^2 - 33(w + 5) + 847 = w^3 - 108w + 432.$$

We have $y^3 + z^3 = -432$ while $y^3z^3 = \frac{108^3}{27} = 46656$. From $0 = t^2 + 432t - 46656 = (t + 216)^2$, we have $y = z = \sqrt[3]{-216} = -6$. Hence the roots of the transformed cubic equation are $y + z = -12$, $y\omega + z\omega^2 = 3 - 3\sqrt{3}i + 3 + 3\sqrt{3}i = 6$ and $y\omega^2 + z\omega = 6$, so that the roots of the original cubic equation are $-7$, 11 and 11.

8. We have $y^3+z^3=12$ while $y^3z^3=-\frac{12^3}{27}=-64$. From $0=t^2-12t-64=(t-16)(t+4)$, we have $y=\sqrt[3]{16}=2(\sqrt[3]{2})$ and $z=\sqrt[3]{-4}=-\sqrt[3]{4}$. Hence the real root of the cubic equation is $y+z=2(\sqrt[3]{2})-\sqrt[3]{4}$.

10. Let $0=x^4-10x^2-20x-16=(x^2+kx+\ell)(x^2-kx+m)=x^4+(m+\ell-k^2)x^2+k(m-\ell)x+m\ell$. Then $m+\ell=k^2-10$, $m-\ell=-\frac{20}{k}$ and $m\ell=-16$. From the first two of these three equations, we have $2m=k^2-10-\frac{20}{k}$ and $2\ell=k^2-10+\frac{20}{k}$. Using the third equation, we have $-64=4m\ell=(k^2-10-\frac{20}{k})(k^2-10+\frac{20}{k})=(k^2-10)^2-(\frac{20}{k})^2=k^4-20k^2+100-\frac{400}{k^2}$. This is equivalent to $k^6-20k^4+164k^2-400=0$. By inspection, $k=2$ is a root. Hence $m=-8$ and $\ell=2$. From $x^2+2x+2=0$, we have $x=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$. From $0=x^2-2x-8=(x-4)(x+2)$, we have $x=4$ or $-2$.

12. We have

$$\begin{aligned}(ax+b)^2 &= x^4+2x^3+(a^2-7)x^2+2(ab-4)x+(b^2+12)\\ &= (x^2+x+k)^2\\ &= x^4+2x^3+(2k+1)x^2+2kx+k^2.\end{aligned}$$

Hence $a^2=2k+8$, $ab=k+4$ and $b^2=k^2-12$. It follows that

$$(k+4)^2=(ab)^2=a^2b^2=(2k+8)(k^2-12).$$

This is equivalent to $2k^3+7k^2-32k-112=0$. By inspection, $k=4$ is a root. Since $ab=8$, $a$ and $b$ are of the same sign. Hence $a=4$ and $b=2$. From $x^2+x+4=4x+2$, we have $0=x^2-3x+2=(x-1)(x-2)$ so that $x=1$ or 2. From $x^2+x+4=-4x-2$, we have $0=x^2+5x+6=(x+2)(x+3)$ so that $x=-2$ or $-3$.

14. We have

$$\begin{aligned}(ax+b)^2 &= x^4-2x^3+(a^2-12)x^2+2(ab+5)x+(b^2+3)\\ &= (x^2-x+k)^2\\ &= x^4-2x^3+(2k+1)x^2-2kx+k^2.\end{aligned}$$

Hence $a^2=2k+13$, $ab=-k-5$ and $b^2=k^2-3$. It follows that

$$(-k-5)^2=(ab)^2=a^2b^2=(2k+13)(k^2-3).$$

This is equivalent to $k^3+6k^2-8k-32=0$. By inspection, $k=-2$ is a root. Since $ab=-3$, $a$ and $b$ are of opposite signs. Hence $a=3$ and $b=-1$. From $x^2-x-2=3x-1$, we have $x^2-4x-1=0$ so that $x=\frac{4\pm\sqrt{16+4}}{2}=2\pm\sqrt{5}$. From $x^2-x-2=-3x+1$, we have $0=x^2+2x-3=(x-1)(x+3)$ so that $x=1$ or $-3$.

16. We have

$$\begin{aligned} 0 &= x^5 - 6x^4 - 17x^3 + 17x^2 + 6x - 1 \\ &= (x^5 - 1) - 6x(x^3 - 1) - 17x^2(x - 1) \\ &= (x - 1)(x^4 + x^3 + x^2 + x^1 - 6(x^2 + x + 1) - 17x^2) \\ &= (x - 1)(x^4 - 5x^3 - 22x^2 - 5x + 1). \end{aligned}$$

Hence one of the roots is 1. Let $y = x + \frac{1}{x}$. Then $y^2 = x^2 + \frac{1}{x^2} + 2$. Hence

$$0 = x^2 - 5x - 22 - \frac{5}{x} + \frac{1}{x^2} = y^2 - 5y - 24 = (y - 8)(y + 3).$$

From $x + \frac{1}{x} = 8$, we have $x^2 - 8x + 1 = 0$ so that $x = \frac{8 \pm \sqrt{64-4}}{2} = 4 \pm \sqrt{15}$. From $x + \frac{1}{x} = -3$, we have $x^2 + 3x + 1 = 0$ so that $x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$.

18. (a) Comparing $x^4 = (x^2 + ax + b) = x^4 - +2ax^3 + (a^2 + 2b)x^2 + 2abx + b^2$ with $0 = 2ax^3 + 2aqx + 2ar$, we have $a^2 + b = 0$, $q = b$ nd $2ar = b^2$. Hence $q^4 = b^4 = 4a^2r^2 = 4(-2b)r^2 = -8qr^2$ so that $q^3 + 8r^2 = 0$.

    (b) In $8x^3 - 36x + 27 = 0$, $q = -\frac{9}{2}$ while $r = \frac{27}{8}$, and indeed $q^3 + 8r^2 = 0$. Thus we may take $b = q = -\frac{9}{2}$ and $a = \frac{b^2}{2r} = 3$ so that $x^4 = (x^2 + 3x - \frac{9}{2})^2$. From $x^2 = x^2 + 3x - \frac{9}{2}$, we have $x = \frac{3}{2}$. From $-x^2 = x^2 + 3x - \frac{9}{2}$, we have $4x^2 + 6x - 9 = 0$ so that $x = \frac{-6 \pm \sqrt{36+144}}{8} = \frac{-3 \pm 3\sqrt{5}}{4}$.

20. Let the roots be $\alpha$, $\alpha$ and $\beta$. Then $2\alpha + \beta = -\frac{3b}{a}$, $\alpha^2 + 2\alpha\beta = \frac{3c}{a}$ and $\alpha^2\beta = -\frac{d}{a}$. Now $\frac{9}{a^2}(bc - ad) = \frac{3b}{a} \cdot \frac{3c}{a} - 9\frac{d}{a} = (-2\alpha - \beta)(\alpha^2 + 2\alpha\beta) + 9\alpha^2\beta = 2\alpha(-\alpha^2 + 2\alpha\beta - \beta^2)$ while $\frac{18}{a^2}(ac - b^2) = 6\frac{3c}{a} - 2(-\frac{3b}{a})^2 = 6(\alpha^2 + 2\alpha\beta) - 2(2\alpha + \beta)^2 = 2(-\alpha^2 + 2\alpha\beta - \beta^2)$. Division yields $\alpha = \frac{bc - ad}{2(ac - b^2)}$.

22. Since all coefficients are rational, irrational roots appear in conjugate pairs, so that $-2 - \sqrt{6}$ is also a root. Hence the original polynomial is divisible by

$$(x - (-2 + \sqrt{6}))(x - (-2 - \sqrt{6})) = (x + 2)^2 - (\sqrt{6})^2 = x^2 + 4x - 2.$$

The quotient turns out to be $x^4 - 4x^3 + 8x - 4$. now

$$\begin{aligned} (ax + b)^2 &= x^4 - 4x^3 + a^2x^2 + 2(ab + 4)x + (b^2 - 4) \\ &= (x^2 - 2x + k)^2 \\ &= x^4 - 4x^3 + (2k + 4)x^2 - 4kx + k^2. \end{aligned}$$

Hence $a^2 = 2k + 4$, $ab = -2k - 4$ and $b^2 = k^2 + 4$. It follows that

$$(-2k - 4)^2 = (ab)^2 = a^2b^2 = (2k + 4)(k^2 + 4).$$

This is equivalent to $0 = k^3 - 4k^2 = k(k - 2)(k + 2)$. We take the root $k = 0$. Since $ab = -4$, $a$ and $b$ are of opposite signs. Hence $a = 2$

and $b=-2$. From $x^2-2x=2x-2$, we have $x^2-4x+2=0$ so that $x=\frac{4\pm\sqrt{16-8}}{2}=2\pm\sqrt{2}$. From $x^2-2x=-2x+2$, we have $x^2=2$ so that $x=\pm\sqrt{2}$.

24. Let the roots be $\alpha$, $\beta$, $\gamma$ and $\delta$. Then $\alpha+\beta+\gamma+\delta=p$, $\alpha\beta\gamma\delta=s$, $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=q$ and $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=r$.

    (a) Suppose $\alpha+\beta=\gamma+\delta=\frac{p}{2}$. Then $\alpha\beta+\gamma\delta=\alpha\beta\frac{2(\gamma+\delta)}{p}+\gamma\delta\frac{2(\alpha+\beta)}{p}=\frac{2r}{p}$. Also, $\alpha\beta+\gamma\delta=q-\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta=q-\alpha(\gamma+\delta)-\beta(\gamma+\delta)=q-(\alpha+\beta)(\gamma+\delta)=q-\frac{p^2}{4}$. Hence $\frac{2r}{p}=q-\frac{p^2}{4}$ or $p^3-4pq+8r=0$.

    (b) Suppose $\alpha\beta=\gamma\delta=\sqrt{s}$ instead. Then $\frac{r}{\sqrt{s}}=\frac{\alpha\beta\gamma+\alpha\beta\delta}{\alpha\beta}+\frac{\alpha\gamma\delta+\beta\gamma\delta}{\gamma\delta}=\gamma+\delta+\alpha+\beta=p$ so that $p^2s=r^2$.

26. Let the product of these two roots be $k$. Then the original polynomial is divisible by $x^2-5x+k$.

$$
\begin{array}{rcccccccc}
 & & & & x^3 & +5x^2 & +(25-k)x & +5(25-2k) \\ \cline{3-8}
x^2-5x+k & ) & x^5 & & & & -409x & +285 \\
 & & x^5 & -5x^4 & +kx^3 & & & \\ \cline{3-8}
 & & & 5x^4 & -kx^3 & & & \\
 & & & 5x^4 & -25x^3 & +5kx^2 & & \\ \cline{4-8}
 & & & & (25-k)x^3 & -5kx & -409x & \\
 & & & & (25-k)x^3 & -5(25-k)x & k(25-k)x & \\ \cline{5-8}
 & & & & & 5(25-2k)x^2 & -(409+25k-k^2)x & +285 \\
 & & & & & 5(25-2k)x^2 & -25(25-2k)x & +5(25-2k)k. \\ \cline{6-8}
\end{array}
$$

From $0=k^2-75k+216=(k-3)(k-72)$ and $0=10k^2-125k+285=5(k-3)(2k-19)$, we see that $k=3$. From $x^2-5x+3=0$, we have $x=\frac{5\pm\sqrt{25-12}}{2}=\frac{5\pm\sqrt{13}}{2}$.

28. Let the product of these roots be $k$. Then the original polynomial is divisible by $x^2-4x+k$. Using Horner's method, we have

$$
\begin{array}{rccccccc}
 & & & & x^2 & -4x & -(k-5) \\ \cline{3-7}
x^2-4x+k & ) & x^4 & -8x^3 & +21x^2 & -20x & +5 \\
 & & x^4 & -4x^3 & +kx^2 & & \\ \cline{3-7}
 & & & -4x^3 & -(k-21)x^2 & -20x & \\
 & & & -4x^3 & +16x^2 & -4kx & \\ \cline{4-7}
 & & & & -(k-5)x^2 & +4(k-5)x & +5 \\
 & & & & -(k-5)x^2 & +4(k-5)x & -k(k-5) \\ \cline{5-7}
\end{array}
$$

From $0=5+k(k-5)=k^2-5k+5$, we have $k=\frac{5\pm\sqrt{25-20}}{2}=\frac{5\pm\sqrt{5}}{2}$. Taking $k=\frac{5+\sqrt{5}}{2}$, we have $x^2-4x+\frac{5+\sqrt{5}}{2}=0$ so that $x=\frac{4\pm\sqrt{16-2(5+\sqrt{5})}}{2}=\frac{4\pm\sqrt{6-2\sqrt{5}}}{2}=\frac{4\pm(\sqrt{5}-1)}{2}$. This yields the roots $\frac{3+\sqrt{5}}{2}$ and $\frac{5-\sqrt{5}}{2}$. If we take $k=\frac{5-\sqrt{5}}{2}$ instead, we get the roots $\frac{3-\sqrt{5}}{2}$ and $\frac{5+\sqrt{5}}{2}$.

## Answers to Odd-numbered Exercises IX

1. $\frac{-5\pm\sqrt{3}i}{2}$.

3. $4, -2 \pm \sqrt{3}i$.

5. $-\frac{1}{4}, \frac{2\pm\sqrt{3}i}{2}$.

7. $-\frac{1}{2}, \frac{-1\pm\sqrt{3}i}{2}$.

9. $4, -1, \frac{-3\pm\sqrt{31}i}{2}$.

11. $\pm 1, -4 \pm \sqrt{6}$.

13. $1 \pm \sqrt{2}, -1 \pm i$.

15. $2, 2, -\frac{1}{2}, -\frac{1}{2}$.

17. $-4, -4, -4, 3$.

23. $s^3y^4 + qs(1-s)^2y^2 + r(1-s)^3y + (1-s)^4 = 0$.

25. $2 \pm \sqrt{3}$.

## Commentary IX

This section was the climax of the course, it divided the class into those who acquired the skill of solving numerical cubics and quartics with numbers that work out well and those who enjoyed the experience of witnessing their exposition. About half of the class did not attempt this question on the final exam. To be fair, the steps can be difficult to remember even for those who follow them well.

In Part 2 we present the material historically backwards in order to start with Descartes' solution to the quartic. The easiest to follow this method serves as the model to the two harder methods that follow. The cubic and quartic are given separate chapters.

An extensive historical profile is provided to lend weight to the subject, and the exercises that follow the examples directly are presented first before a selection of exercises that make use of material from both chapters in addition to previous chapters.

# Part 2

# PREFACE

Why write (or read) the preface to a book? The preface should tell the reader *why* to read the book and give some suggestions on *how* to read it.

This book is about the theory of equations. It is based on some chapters from the classic book by Hall and Knight called Higher Algebra (1887). Their book sold hundreds of thousands of copies all around the world. This book is a partial tribute to that work and a different approach to the subject than has become standard. If the present work seems terse or, notably, without figures, this style may be interpreted as loyalty to the original and an intentional departure from modern 'multimedia' texts.

The highlights of the present work are the solutions to the general quadratic, cubic, and quartic equations; a number of theorems concerning them; and the Arithmetic Mean-Geometric Mean Inequality. Although the quadratic formula is still commonly taught, most modern students are unacquainted with the solutions to the cubic and quartic and will be surprized to see how they too can be solved in general.

Several audiences have been in mind during the writing of this book. Teachers of middle school mathematics are one group of readers that can benefit from this book. This course provides an opportunity to develop algebraic skills that are relevant to the topics taught in school, but that are more advanced and thus afford the teacher greater perspective. The text has been tested with precisely this group of students. The book also has broad appeal to recreational readers of mathematics and may be an ideal resource for advanced middle or senior school students.

Each chapter has a large set of exercises. These form an integral part of the book and should be attempted to clarify and practice the topics from the chapter. Solutions have been provided for the even-numbered exercises and answers appear for the odd-numbered exercises. As one progresses through the text one should take the time to consider how the components fit together and question the significance of each result.

*Hall and Knight's Higher Algebra: The Theory of Equations* was adapted by the present authors over the course of teaching two classes of Math 164: Higher Algebra at the University of Alberta between 2001 and 2003. The students were middle school teachers who were upgrading their mathematics background in response to an initiative for better qualified math teachers. Material similar to what is presented here was taught in 12 weekly lectures of $2\frac{1}{2}$ hours.

# CHAPTER 0 — PRELIMINARIES

1. Number Systems
2. Products of Unknowns
3. Sums, Differences, and Products
4. Factoring Trinomials
5. Factoring Special Binomials
6. Synthetic Division and Horner's Method

## Number Systems

While algebra concerns the insights that can be made by the manipulation of letters in place of numbers it will be important to clarify the system of numbers considered. On the other hand it is the solutions to different algebraic equations that require subsequent extensions of the number system.

The *natural* numbers are the the so called 'counting' numbers $1, 2, 3, 4, 5, \ldots$ The natural numbers form an infinite set with no biggest number. The natural numbers are sufficient to solve equations of the type $a+x=b$ where $b$ is greater than $a$, with $a, b$ both themselves natural. Since the sum of any two natural numbers is a natural number, we say that the natural numbers are closed under addition.

**Example 1.**
Solve $3+x=7$.

If we wish to consider the equation $a+x=a$ we must include the number zero. The set of natural numbers and zero is called the *whole* numbers or, in light of what follows, the non-negative integers. The *integers* are zero and the natural numbers assigned either a positive or negative sign. It should be clear that the integers include the set of natural numbers. Since the sum or difference of any two integers is an integer, we say that the integers are closed under addition and subtraction. The integers are sufficient to solve all equations of the form $a+x=b$, where $a$ and $b$ are themselves integers.

If we demand further that the set be closed under division, we shall require the extension to the *rational* numbers. The rationals are all numbers in the form $\frac{m}{n}$, $m, n$ integers, $n \neq 0$. The rationals are closed under addition, subtraction, multiplication, and division and are sufficient to solve all equations of the form $ax+b=0$ where $a, b$ are themselves rational. The numbers $a, b$ are called *coefficients* and in this text we shall always consider equations with rational coefficients.

**Example 2.**
Solve $5x + 20 = 0$ and $\frac{4}{5}x + 24 = 0$

Although the rational numbers appear to 'do it all' we need look no further than the equation $x^2 = 2$ for an example that cannot be solved over the rationals. In fact, the solutions $\pm\sqrt{2}$ are *irrational* numbers. On a calculator, irrationals look like endless decimals with no pattern. The set of irrationals and rationals together form the *real* numbers. If you like, the reals are the rationals and everything in between.

For many purposes the real numbers are 'everything' and need not be further extended, but we shall see that there are equations that cannot be solved over the reals. This observation will lead to the definition of a new number system called the *complex* numbers. The complex numbers are the reals and some new *imaginary* numbers.

**Example 3.**
Draw a picture showing the inclusion of these number systems inside each other. Think of examples of each type.

## Products of Unknowns

Often we shall have information about the product of unknown algebraical quantities and wish to obtain information about them individually.

**Theorem 1**
If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.

**Proof:**

Suppose that $a$ is not zero and check if it is possible that $b$ be nonzero also.

$$a \cdot b = 0 = a \cdot 0$$

therefore by cancellation, $b = 0$. So if $a$ is nonzero $b$ must be zero. Similarly, if $b$ is nonzero $a$ must be zero.

**Example 1**
Suppose $(x - 1)(x - 2) = 0$. What values can $x$ take?

**Solution:**

Since the product of two quantities is zero, one of them must be zero (Theorem 1). So either $x - 1 = 0$ or $x - 2 = 0$. Therefore $x = 1$ or $x = 2$.

**Theorem 2**
If $a \cdot b < 0$, then one of $a$ and $b$ is positive and the other is negative.

**Proof:**

If $a$ and $b$ were of like signs their product would be positive, so they must be of opposite signs.

# Sum, Difference and Product of Two Numbers

**Sum and Difference.**
It is a standard result that given the sum and the difference of two numbers, they can be determined as follows. Let the numbers be $a$ and $b$ with $a > b$, their sum be $s$ and their difference be $d$. Then $a + b = s$ and $a - b = d$. Addition yields $2a = s + d$ so that $a = \frac{s+d}{2}$. Subtraction yields $2b = s - d$ so that $b = \frac{s-d}{2}$.

**Sum and Product.**
Given the sum and the product of two numbers, their difference can be determined as follows. Let the numbers be $a$ and $b$ with $a > b$, their sum be $s$ and their product be $p$. Then

$$\begin{aligned}(a-b)^2 &= a^2 - 2ab + b^2\\ &= a^2 + 2ab + b^2 - 4ab\\ &= (a+b)^2 - 4ab\\ &= s^2 - 4p.\end{aligned}$$

It follows that $a - b = \sqrt{s^2 - 4p}$.

**Difference and Product.**
Given the difference and the product of two numbers, their sum can be determined as follows. Let the numbers be $a$ and $b$ with $a + b > 0$, their difference be $d$ and their product be $p$. Then

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2\\ &= a^2 - 2ab + b^2 + 4ab\\ &= (a+b)^2 + 4ab\\ &= s^2 + 4p.\end{aligned}$$

It follows that $a + b = \sqrt{s^2 + 4p}$.

**Examples.**

(a) Determine two numbers whose sum is $-2$ and whose product is $-8$.

(b) Determine two numbers whose difference is 5 and whose product is $-6$, given that their sum is negative.

**Solution:**

(a) Let the numbers be $a$ and $b$ with $a > b$. Then $a - b = \sqrt{(-2)^2 - 4(-8)} = 6$. Hence $a = \frac{-2+6}{2} = 2$ and $b = \frac{-2-6}{2} = -4$.

(b) Let the numbers be $a$ and $b$ with $a > b$. Then $a + b = -\sqrt{5^2 + 4(-6)} = -1$. Hence $a = \frac{-1+5}{2} = 2$ and $b = \frac{-1-5}{2} = -3$.

## Factoring Trinomials

The student is advised to review the way in which, in forming the product of two binomials, the coefficients of the different terms combine so as to give a trinomial result. For instance,

$$
\begin{aligned}
(x+5)(x+3) &= x^2+8x+15, & (1)\\
(x-5)(x-3) &= x^2-8x+15, & (2)\\
(x+5)(x-3) &= x^2+2x-15, & (3)\\
(x-5)(x+3) &= x^2-2x-15. & (4)
\end{aligned}
$$

We now propose to consider the converse problem: namely, the Factorization of a trinomial expression similar to those which occur on the right-hand side of the above identities.

By examining the above results, we notice that:

1. The first term of both the factors is $x$.
2. The product of the second terms of the two factors is equal to the *third term* of the trinomial; e.g. in (2) above we see that 15 is the product of $-5$ and $-3$; while in (3) $-15$ is the product of $+5$ and $-3$.
3. The *algebraic sum* of the second terms of the two factors is equal to the *coefficient* of $x$ in the trinomial; e.g. in (4) the sum of $-5$ and $+3$ gives $-2$, the coefficient of $x$ in the trinomial.

In applying these laws we will first consider a case where *the third term of the trinomial is positive.*

**Example 1.**
Factorize $x^2+11x+24$.

**Solution:**

The second terms of the factors must be such that their product is $+24$, and their sum $+11$. It is clear that they must be $+8$ and $+3$. Hence $x^2+11x+24 = (x+8)(x+3)$.

**Example 2.**
Factorize $x^2-10x+24$.

**Solution:**

The second terms of the factors must be such that their product is $+24$, and their sum $-10$. Hence they must *both be negative*, and it is easy to see that they must be $-6$ and $-4$. Hence $x^2-10x+24 = (x-6)(x-4)$.

**Example 3.**
Factorize $x^2-18x+81$.

**Solution:**

We have $x^2 - 18x + 81 = (x - 9)(x - 9) = (x - 9)^2$.

**Example 4.**
Factorize $x^4 + 10x^2 + 25$.

**Solution:**

We have $x^4 + 10x^2 + 25 = (x^2 + 5)(x^2 + 5) = (x^2 + 5)^2$.

**Example 5.**
Factorize $x^2 - 11ax + 10a^2$.

**Solution:**

The second terms of the factors must be such that their product is $+10a^2$, and their sum $-11a$. Hence they must be $-10a$ and $-a$, so that $x^2 - 11ax + 10a^2 = (x - 10a)(x - a)$.

NOTE. In examples of this kind the student should always verify his results, by *mentally* forming the product of the factors he has chosen.

Next consider a case where *the third term of the trinomial is negative.*

**Example 6.**
Factorize $x^2 + 2x - 35$.

**Solution:**

The second terms of the factors must be such that their product is $-35$, and their *algebraic sum* $+2$. Hence they must have *opposite* signs, and the greater of them must be *positive* in order to give its sign to their sum. The required terms are therefore $+7$ and $-5$. Hence $x^2 + 2x - 35 = (x + 7)(x - 5)$.

**Example 7.**
Factorize $x^2 - 3x - 54$.

**Solution:**

The second terms of the factors must be such that their product is $-54$, and their *algebraic sum* $-3$. Hence they must have *opposite* signs, and the greater of them must be *negative* in order to give its sign to their sum. The required terms are therefore $-9$ and $+6$. Hence $x^2 - 3x - 54 = (x - 9)(x + 6)$.

Remembering that in these cases the numerical quantities *must have opposite signs.* If preferred, the following method may be adopted.

**Example 8.**
Factorize $x^2y^2 + 23xy - 420$.

**Solution:**

Find two numbers whose product is 420, and whose *difference* is 23. These are 35 and 12; hence inserting the signs so that the positive may predominate, we have3 $x^2y^2 + 23xy - 420 = (xy + 35)(xy - 12)$.

We proceed now to the resolution into factors of trinomial expression *when the coefficient of the highest power is not unity.*

Again, we may write down the following results:

$$\begin{aligned}(3x+2)(x+4) &= 3x^2+14x+8,\\ (3x-2)(x-4) &= 3x^2-14x+8,\\ (3x+2)(x-4) &= 3x^2-10x-8,\\ (3x-2)(x+4) &= 3x^2+10x-8.\end{aligned}$$

The converse problem presents more difficulty than the cases we have yet considered.

Before endeavoring to give a general method of procedure, it will be worth while to examine in detail two of the identities given above.

Consider the result $3x^2-14x+8=(3x-2)(x-4)$. The first term $3x^2$ is the product of $3x$ and $x$. The third term $+8$ is the product of $-2$ and $-4$. The middle term $-14x$ is the result of adding together the two products $3x(-4)$ and $x(-2)$.

Again, consider the result $3x^2-10x-8=(3x+2)(x-4)$. The first term $3x^2$ is the product of $3x$ and $x$. The third term $-8$ is the product of $+2$ and $-4$. The middle term $-10x$ is the result of adding together the two products $3x(-4)$ and $x(2)$ — and its sign is negative because the greater of these two products is negative.

The beginner will frequently find that it is not easy to select the proper factors at the first trial. Practice alone will enable him to detect at a glance whether any pair he has chosen will combine so as to give the correct coefficients of the expression to be factored.

**Example 9.**
Factorize $7x^2-19x-6$.

**Solution:**

Write down $(7x\ \ 3)(x\ \ 2)$ for a first trial, noticing that 3 and 2 must have opposite signs. These factors give $7x^2$ and $-6$ for the first and third terms. But since $7\times2-3\times1=11$, the combination falls to give the correct coefficient of the middle term. Next try $(7x\ 2)(x\ 3)$. Since $7\times3-2\times1=19$, these factors will be correct if we insert the signs so that the negative shall predominate. Thus $7x^2-19x-6=(7x+2)(x-3)$. The student should verify this by mental multiplication.

In actual work it will not be necessary to put down all these steps at length. The student will soon find that the different cases may be rapidly reviewed, and the unsuitable combinations rejected at once.

It is especially important to pay attention to the two following two hints :

1. If the third term of the trinomial is positive, then the second terms of its factors have both the same sign, and this sign is the same as that of the middle term of the trinomial.

2. If the third term of the trinomial is negative, then the second terms of its factors have opposite signs.

**Example 10.**
Factorize $14x^2 + 29x - 15$ and $14x^2 - 29x - 15$.

**Solution:**

In each case we may write down $(7x \quad 3)(2x \quad 5)$ as a first trial, noticing that 3 and 5 must have opposite signs. And since $7 \times 5 - 3 \times 2 = 29$, we have only now to insert the proper signs in each factor. In the first expression, the positive sign must predominate; in the second expression, the negative sign must dominate. Therefore $14x^2 + 29x - 15 = (7x - 3)(2x + 5)$ and $14x^2 - 29x - 15 = (7x + 3)(2x - 5)$.

**Example 11.**
Factorize $5x^2 + 17x + 6$ and $5x^2 - 17x + 6$.

**Solution:**

In the first expression, we notice that the factors which give 6 are both positive. In the second expression, they are negative. And therefore for the first expression, we may write $(5x + \quad)(x + \quad)$, and for the second expression, we may write $(5x - \quad)(x - \quad)$. And, since $5 \times 3 + 1 \times 2 = 17$, we see that $5x^2 + 17x + 6 = (5x + 2)(x + 3)$ and $5x^2 - 17x + 6 = (5x - 2)(x - 3)$.

NOTE: in each expression the third term 6 also admits of factors 6 and 1; but this is one of the cases referred to above which the student would reject at once as unsuitable.

**Example 12.**
Factorize $9x^2 - 48xy + 64y^2$.

**Solution:**

We have $9x^2 - 48xy + 64y^2 = (3x - 8y)(3x - 8y) = (3x - 8y)^2$.

**Example 13.**
Factorize $6 + 7x - 5x^2$.

**Solution:**

We have $6 + 7x - 5x^2 = (3 + 5x)(2 - x)$.

**Example 14.**
Factorize $28x^4y + 64x^3y - 60x^2y$.

**Solution:**

We have $28x^4y + 64x^3y - 60x^2y = 4x^2y(7x^2 + 16x - 15) = 4x^2y(7x - 5)(x + 3)$.

## Factoring Special Binomials

The following identities may be directly verified. Their application will be made at various points in the text.

| | | | |
|---|---|---|---|
| Difference of Squares | $a^2 - b^2$ | $=$ | $(a+b)(a-b)$ ; |
| Difference of Cubes | $a^3 - b^3$ | $=$ | $(a-b)(a^2+ab+b^2)$ ; |
| Sum of Cubes | $a^3 + b^3$ | $=$ | $(a+b)(a^2-ab+b^2)$ ; |

These identities are small cases of the result for the sum or difference of $n^{th}$ powers, however they will suffice.

**Example 1.**
Simplify $\frac{27x^3-8}{3x-2}$.

**Solution:**

We have $\frac{27x^3-8}{3x-2} = \frac{(3x-2)(9x^2+6x+4)}{3x-2} = 9x^2+6x+4$.

**Example 2.**
Factor completely $x^4 - 81$.

**Solution:**

We have $x^4 - 81 = (x^2+9)(x^2-9) = (x^2+9)(x+3)(x-3)$

## Synthetic Division and Horner's Method

**Example 1.**
Find the quotient and remainder when $3x^7 - x^6 + 31x^4 + 21x + 5$ is divided by $x+2$.

**First Solution:**

$$
\begin{array}{rrrrrrrrr}
 & & 3x^6 & -7x^5 & 14x^4 & 3x^3 & -6x^2 & +12x & -3 \\
\cline{2-9}
x+2 \;) & 3x^7 & -x^6 & & +31x^4 & & & +21x & +5 \\
 & 3x^7 & +6x^6 & & & & & & \\
\cline{2-4}
 & & -7x^6 & & & & & & \\
 & & -7x^6 & -14x^5 & & & & & \\
\cline{3-5}
 & & & 14x^5 & +31x^4 & & & & \\
 & & & 14x^5 & +28x^4 & & & & \\
\cline{4-6}
 & & & & 3x^4 & & & & \\
 & & & & 3x^4 & +6x^3 & & & \\
\cline{5-7}
 & & & & & -6x^3 & & & \\
 & & & & & -6x^3 & -12x^2 & & \\
\cline{6-8}
 & & & & & & 12x^2 & +21x & \\
 & & & & & & 12x^2 & +24x & \\
\cline{7-9}
 & & & & & & & -3x & +5 \\
 & & & & & & & -3x & -6 \\
\cline{8-9}
 & & & & & & & & 11
\end{array}
$$

The quotient is $3x^6 - 7x^5 + 14x^4 + 3x^3 - 6x^2 + 12x - 3$ and the remainder is 11.

**Second Solution:**

$$\begin{array}{r|rrrrrrrr} -2 & 3 & -1 & 0 & 31 & 0 & 0 & 21 & 5 \\ & & -6 & 14 & -28 & -6 & 12 & -24 & 6 \\ \hline & 3 & -7 & 14 & 3 & -6 & 12 & -3 & 11 \end{array}$$

The quotient is $3x^6 - 7x^5 + 14x^4 + 3x^3 - 6x^2 + 12x - 3$ and the remainder is 11.

The first solution uses standard long division while the second solution uses **synthetic division**. The latter is a combination of two labour-saving devices. The first is lining up the terms properly so that all references to powers of $x$ may be omitted. We also omit the leading $x$ in the divider $x + 2$. The second is to change the sign of the constant term so that it is now $-2$. This way, instead of doing a lot of subtractions, we will be doing a lot of additions.

Here is the procedure. The coefficients are written down in order in the first row, with 0 inserted for missing terms. In the first column to the right of the vertical line, the 3 is simply brought down to the third row. Then we multiply 3 by $-2$ and enter $-6$ in the second row of the next column. An addition is performed, yielding $-7$ in the third row. This is in turn multiplied by $-2$ to yield 14 in the second row of the next column, and so on. At the end, the last term in the third row is the remainder while the others are the coefficients of the quotient in order, starting with one power lower than the dividend.

**Example 2.**
Find the quotient and remainder when $4x^5 - 4x^4 + 3x^3 - 7x^2 - x + 1$ is divided by $2x - 1$.

**Solution:**

$$\begin{array}{r|rrrrrr} \frac{1}{2} & 4 & -4 & 3 & -7 & -1 & 1 \\ & & 2 & -1 & 1 & -3 & -2 \\ \hline & 4 & -2 & 2 & -6 & -4 & -1 \end{array}$$

We first divide the divider $2x - 1$ by its leading coefficient to obtain $x - \frac{1}{2}$. Eventually, we divide the quotient by the same coefficient to obtain $2x^4 - x^3 + x^2 - 3x - 2$ while the remainder $-1$ is unaffected.

**Example 3.**
Find the quotient and remainder when $3x^5 - 8x^4 - 5x^3 + 26x^2 - 33x + 26$ is divided by $x^3 - 2x^2 - 4x + 8$.

**Solution:**

$$\begin{array}{r|rrrrrr} 2,4,-8 & 3 & -8 & -5 & 26 & -33 & 26 \\ & & 6 & 12 & -24 & & \\ & & & -4 & -8 & 16 & \\ & & & & 6 & 12 & -24 \\ \hline & 3 & -2 & 3 & 0 & -5 & 2 \end{array}$$

The quotient is $3x^2 - 2x + 3$ and the remainder is $-5x + 2$.

The solution in Example 3 uses **Horner's Method** which is a natural generalization of synthetic division to the case when the divider is a non-linear polynomial. The leading $x^3$ in the divider $x^3 - 2x^2 - 4x + 8$ is omitted, and the coefficients of the remaining terms change signs. In the first column to the right of the vertical line, the 3 is brought down to the fifth row. We then multiply it successively by 2, 4 and 18, and enter the results in the second row shifted one column to the right. We now perform an addition in the next column which yields $-2$ in the fifth row. This is in turn multiplied by 2, 4 and $-8$, and the results entered in the third row, and so on. The left side of the fifth row is the quotient and the right side is the remainder. The remainder has the same number of terms as the number of coefficients to the left of the vertical line.

# CHAPTER I — QUADRATIC EQUATIONS

## Polynomial Functions.

The simplest class of functions is the class of *polynomial* functions. Given any input value the function determines an output. For example, the function $f(x) = x^3 - x + 1$ returns, for the input 3, the output 25. Moreover, we write $f(3) = 25$ and read '$f$ of 3 equals 25'.

**Example 1.**
Evaluate $f(x) = x^2 - 6x + 1$ for $x = 2$ and $x = -2$.

**Solution:**

$$f(2) = (2)^2 - 6(2) + 1 = -7 \text{ and } f(-2) = (-2)^2 - 6(-2) + 1 = 17.$$

## Rational Functions.

The class of *rational funtions* is made up of functions that have for their numerator and denominator a polynomial function. For example, $r(x) = \frac{4x+1}{x^2-4}$ is a rational function. Due to the possibility of dividing by zero for certain inputs we sometimes must enforce a restriction. The set of valid inputs is called the *domain* of the function. For the function $r$ (above) the maximum domain is 'all real values excluding 2 and $-2$'. It is worth noting that polynomial functions never need a restriction on their domain; the maximum domain for each polynomial function is 'all reals'.

A concept related to domain is *range*. The range of a function is the set in which the outputs are elements. The values achieved by rational functions is the subject of part of chapter VII.

## Equations and Roots.

The '=' symbol is used in $f(x) = 4x - 12$ in what can be called the defining sense. Having defined $f$ we can refer to the algebraic expression by its name. The same symbol is used, however, also in a different sense. An *equation* is a mathematical sentence which sets equal two expressions. Usually we shall be interested in finding the values of the variable for which this equation holds. In particular, if a function is set equal to zero, we shall call the solutions the *roots* of the funtion.

**Example 2.**
Find the root of the function $f(x) = 4x - 12$.

**Solution:**

Finding the root is equivalent to solving $f(x) = 0$, hence $4x - 12 = 0$. Adding 12 to both sides and dividing by 4 we have $x = 3$. As a check we see that $f(3) = 0$.

Different kinds of functions may have two or more roots. This chapter will focus on quadratic equations (polynomials of degree two) whereas chapters V and VI will describe quartic and cubic equations (polynomials of degree four and three).

### Quadratic Equations.

The most basic quadratic equation is $x^2 - 1 = 0$. We have $x^2 = 1$, and thus $x = \pm 1$. Therefore we have two roots this time. Related to this example is say, $9x^2 - 4 = 0$. Here we have $x^2 = \frac{4}{9}$, and thus $x = \pm\frac{2}{3}$. The method is to isolate the unknown and take square roots. The example $x^2 + 6x + 1 = 0$ presents a problem because the unknown appears twice. In this case we must apply special artifice to remove the second term. The method is called completing the square.

### Completing the Square.

By taking half of the second coefficient, in the case of the example above half of 6, we can write $(x + 3)^2$. This expression, if expanded, yields the correct first and second terms. It then remains to correct the constant term. One can verify that $(x + 3)^2 - 8$ is the same as $x^2 + 6x + 1$.

### Example 3.

Solve $x^2 + 4x - 6 = 0$ by completing the square.

**Solution:**

$0 = x^2 + 4x - 6 = (x + 2)^2 - 4 - 6 = (x + 2)^2 - 10$, so $(x + 2)^2 = 10$ and therefore $x = -2 \pm \sqrt{10}$.

Since this process is general, we can do it with letters in place of numerical coefficients. This is one way to prove a central result: The Quadratic Formula.

### Theorem - The Quadratic Formula.

The equation $ax^2 + bx + c = 0$ has roots $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

**Proof:**

Dividing the equation by $a$ yields $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$. Then the method of completing the square leads to

$$\begin{aligned}
\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} &= 0 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

### Theorem - Classification of roots.

The expression $b^2 - 4ac$ under the radical is called the *discriminant* of the equation $ax^2 + bx + c = 0$ because the nature of its roots may be determined without solving the equation. Let $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$. Then,

(i) If $b^2 - 4ac = 0$, then $\alpha = \beta = -\frac{b}{2a}$.

(ii) If $b^2 - 4ac$ is a positive square, $\alpha$ and $\beta$ are rational and unequal.

(iii) If $b^2 - 4ac$ is a positive non-square, $\alpha$ and $\beta$ are irrational and unequal.

## EXERCISES I

1. Solve by both completing the square and the quadratic formula
   - (a) $x^2 + 4x + 1 = 0$
   - (b) $x^2 + 6x - 3 = 0$
   - (c) $2x^2 + 5x + 1 = 0$
2. If the equation $x^2 - 15 - m(2x - 8) = 0$ has equal roots, find the values of $m$.
3. For what values of $m$ will the equation $x^2 - 2x(1 + 3m) + 7(3 + 2m) = 0$ have equal roots?
4. For what value of $m$ will the equation $\frac{x^2-bx}{ax-c} = \frac{m-1}{m+1}$ have roots equal in magnitude but opposite in sign?
5. Prove that the roots of the following equations are rational:
   - (a) $(a + c - b)x^2 + 2cx + (b + c - a) = 0$;
   - (b) $abc^2x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$.

### Solutions to Even-numbered Exercises I

2. The equation may be rewritten as $x^2 - 2mx + (8m - 15) = 0$. It has equal roots if and only if $(-2m)^2 = 4(8m - 15)$. This reduces to $m^2 - 8m + 15 = 0$. The desired values are $m = \frac{1}{2}(8 \pm \sqrt{8^2 - 4 \times 15}) = 5$ or $3$.

4. The equation may be rewritten as $(m + 1)x^2 - (b(m + 1) + a(m - 1))x + c(m - 1) = 0$. If the roots are $\pm r$, then the equation has the form $0 = (x - r)(x - (-r)) = x^2 - r^2$. Hence $b(m + 1) + a(m - 1) = 0$ so that $m = \frac{a-b}{a+b}$.

### Answers to Odd-numbered Exercises I

1. (a) $-2 \pm \sqrt{3}$; (b) $-3 \pm 2\sqrt{3}$; (c) $\frac{-5\pm\sqrt{17}}{4}$.

3. $m = -\frac{20}{11}$

# CHAPTER II — QUADRATIC EQUATIONS WITH IMAGINARY ROOTS

a. Complex Numbers

b. Square Roots of Complex Numbers

c. Quadraric Roots

d. Quadratic Formula

## Complex Numbers

After discovering the quadratic formula of Chapter I we have good reason to feel comfortable encountering quadratic equations. There are, however, quadratic equations that will require further study, the simplest of which is

$$x^2 + 1 = 0.$$

This innocuous-looking equation quickly leads to problems since it implies that $x^2 = -1$.

Although from the rule of signs it is evident that a negative quantity cannot have a real square root, imaginary quantities represented by symbols of the form $\sqrt{-a}$ and $\sqrt{-1}$ are of frequent occurrence in mathematical investigations, and their use leads to valuable results. We therefore proceed to explain in what sense such roots are to be regarded.

When the quantity under the radical sign is negative, we can no longer consider the symbol $\sqrt{\ }$ as indicating a possible arithmetical operation; but just as $\sqrt{a}$ may be defined as a symbol which obeys the relation $\sqrt{a}\sqrt{a} = a$, so we shall define $\sqrt{-a}$ to be such that $\sqrt{-a}\sqrt{-a} = -a$, and we shall accept the meaning to which this assumpion leads us.

It will be found that this definition will enable us to bring imaginary quantities under the dominion of ordinary algebraical rules, and that through their use results may be obtained which can be relied on with as much certainty as others which depend solely on the use of real quantities.

By definition, $\sqrt{-1}\sqrt{-1} = -1$. Therefore, $\sqrt{a}\sqrt{-1}\sqrt{a}\sqrt{-1} = a(-1)$. That is, $(\sqrt{a}\sqrt{-1})^2 = -a$. Thus the product $\sqrt{a}\sqrt{-1}$ may be regarded as equivalent to the imaginary quantity $\sqrt{-a}$.

It will be convenient to denote $\sqrt{-1}$ by the symbol $i$. The imaginary character of an expression will be denoted by its presence. For instance, $\sqrt{-4} = 2i$ and $\sqrt{-7a^2} = a\sqrt{7}i$.

We shall always consider that, in the absence of any statement to the contrary, of the signs which may be prefixed before a radical the positive sign is to be taken. But in the use of imaginary quantities, there is one point of importance which deserves notice.

Since $(-a)(-b) = ab$, by taking the square root, we have $\sqrt{-a}\sqrt{-b} = \pm\sqrt{ab}$. Thus in forming the product of $\sqrt{-a}$ and $\sqrt{-b}$ it would appear that either of the signs + or − might be placed before $\sqrt{ab}$. This is not the case however, for $\sqrt{-a}\sqrt{-b} = \sqrt{a}i\sqrt{b}i = \sqrt{ab}i^2 = -\sqrt{ab}$.

A number of the form $a + bi$ is called a **complex number**. Here $a$ and $b$ are real numbers, but ~~not~~ necessarily rational. We call $a$ the real part and $b$ the imaginary part of the complex number $a + bi$. A complex number with zero real part is called imaginary.

In dealing with complex numbers, the usual rules or arithmetic apply.

**Addition/Subtraction Rule.**

$$(a + bi) \pm (c + di) = a \pm c + (b \pm d)i.$$

**Multiplication Rule.**

$$(a + bi) \cdot (c + di) = ac - bd + (ad + bc)i.$$

**Lemma.**
If $a + bi = 0$, then $a = 0$ and $b = 0$.

**Proof:**

Suppose $a + bi = 0$. Then $bi = -a$. Hence $-b^2 = a^2$ so that $a^2 + b^2 = 0$. Now $a^2$ and $b^2$ are both non-negative; therefore their sum cannot be zero unless each of them is zero. That is, $a = 0$ and $b = 0$.

**Theorem 1.**
If $a + bi = c + di$, then $a = c$ and $b = d$.

**Proof:**

By transposition, $a - c + (b - d)i = 0$. By the lemma, $a - c = 0$ and $b - d = 0$; that is, $a = c$ and $b = d$.

*Thus in order that two complex numbers be equal it is necessary and sufficient that the real parts be equal and the imaginary parts be equal.*

**Definition.**
When two complex numbers differ only in the sign of the imaginary part, they are said to be *conjugate.*

For instance, $2 - 3i$ is conjugate to $2 + 3i$. In general, $a - bi$ is conjugate to $a + bi$.

**Theorem 2.**
The sum and product of two conjugate complex numbers are both real.

**Proof:**

We have $(a+bi)+(a-bi) = 2a$ and $(a+bi)(a-bi) = a^2-(-b^2) = a^2+b^2$.

**Definition.**
The positive value of the square root of $a^2 + b^2$ is called the *modulus* of each of the conjugate complex numbers $a + bi$ and $a - bi$.

If the denominator of a fraction is of the form $a+bi$, it may be rationalized by multiplying the numerator and the denominator by the conjugate expression $a - bi$.

For instance,

$$\begin{aligned}\frac{c+di}{a+bi} &= \frac{(c+di)(a-bi)}{(a+bi)(a-bi)}\\ &= \frac{ac+bd+(ad-bc)i}{a^2+b^2}\\ &= \left(\frac{ac+bd}{a^2+b^2}\right)+\left(\frac{ad-bc}{a^2+b^2}\right)i.\end{aligned}$$

Thus we see that the sum, difference, product and quotient of two complex numbers is in each case another complex number.

**Example 1.**
Reduce $\frac{(2+3i)^2}{2+i}$ to the form $A + Bi$.

**Solution:**

The given expression is equal to

$$\begin{aligned}\frac{4-9+12i}{2+i} &= \frac{(-5+12i)(2-i)}{(2+i)(2-i)}\\ &= \frac{-10+12+29i}{4+1}\\ &= \frac{2}{5}+\frac{29}{5}i,\end{aligned}$$

which is of the required form.

### Square Roots of Complex Numbers

We wish to find the square root of $a+bi$. Assume that it is equal to $x+yi$, where $x$ and $y$ are real quantities. By squaring, $a + bi = x^2 - y^2 + 2xyi$. Therefore, by equating real and imaginary parts, $x^2 - y^2 = a$ and $2xy = b$. Now, $(x^2+y^2)^2 = (x^2-y^2)^2+(2xy)^2 = a^2+b^2$. So $x^2+y^2 = \sqrt{a^2+b^2}$. From this and $x^2 - y^2 = a$, we obtain $x^2 = \frac{\sqrt{a^2+b^2}+a}{2}$ and $y^2 = \frac{\sqrt{a^2+b^2}-a}{2}$. Finally, $x = \pm\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}}$ and $y = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}$.

**Remark:**
Since $x$ and $y$ are real numbers, $x^2 + y^2$ is positive, and therefore the positive root is chosen in $x^2 + y^2 = \sqrt{a^2 + b^2}$. Also from $2xy = b$, we see that the product $xy$ must have the same sign as $b$; hence $x$ and $y$ must have like signs if $b$ is positive, and unlike signs if $b$ is negative.

**Example 1.**
Find the square root of $-7 - 24i$.

**Solution:**

Assume $\sqrt{-7 - 24i} = x + yi$; then $-7 - 24i = x^2 - y^2 + 2xyi$. Therefore $x^2 - y^2 = -7$ and $2xy = -24$. Now $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2 = 49 + 576 = 625$. Hence $x^2 + y^2 = 25$. From these equations one can see $x^2 = 9$ and $y^2 = 16$ and therefore $x = \pm 3$ and $y = \pm 4$. However, since the product $xy$ is negative we must take $x = 3$ and $y = -4$, or $x = -3$ and $y = 4$. Finally, $\sqrt{-7 - 24i} = \pm(3 - 4i)$.

**Example 2.**
Find the value of $\sqrt{i}$.

**Solution:**

Assume $\sqrt{i} = x + yi$. Then $i = x^2 - y^2 + 2xyi$. Therefore $x^2 - y^2 = 0$ and $2xy = 1$, whence $x = \frac{1}{\sqrt{2}}$ and $y = \frac{1}{\sqrt{2}}$, or $x = -\frac{1}{\sqrt{2}}$ and $y = -\frac{1}{\sqrt{2}}$. Hence $\sqrt{i} = \pm\frac{1}{\sqrt{2}}(1 + i)$.

It is useful to notice the successive powers of $i$; thus $i^1 = i$, $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$. Since each power is obtained by multiplying the one before it by $i$, we see that the results must now recur in a cycle with these four steps.

We have seen that the complex numbers are closed under the four arithmetical operations and square roots.

## Quadratic Roots

We now come to an important result.

**Vieta's Theorem**
If $\alpha$ and $\beta$ are the roots of $ax^2 + bx + c = 0$, then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

**Proof:**

Clearly, $\alpha$ and $\beta$ are the roots of $(x - \alpha)(x - \beta) = 0$. This expands into $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. Comparing with $x^2 + \frac{b}{a}x + \frac{c}{a}$, we have the desired results.

In fact, any quadratic equation may be expressed in the form

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0.$$

So, given the roots, we can easily form an equation.

**Example 1.**
Form an equation whose roots are 3 and $-2$.

**Solution:**

The equation is $(x-3)(x+2)=0$ or $x^2-x+6=0$.

When the given roots are irrational, it is easier to use the following method.

**Example 2.**
Form an equation whoose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

**Solution:**

We have sum of roots $=4$ and product of roots $=1$; so the equation is $x^2-4x+1=0$.

By a method analogous to that used in Example 1, we can form an equation with three or more given roots.

**Example 3.**
Form an equation whose roots are 2, $-3$ and $\frac{7}{5}$.

**Solution:**

The required equation must be satisfied by each of the following suppositions: $x-2=0$, $x+3=0$ and $x-\frac{7}{5}=0$. Therefore the equation must be $(x-2)(x+3)(x-\frac{7}{5})=0$, which may be rewritten as $(x-2)(x+3)(5x-7)=5x^3-2x^2-37x+42=0$.

**Example 4.**
Form the equation whose roots are 0, $\pm a$ and $\frac{c}{b}$.

**Solution:**

The equation has to be satisfied by $x=0$, $x=a$, $x=-a$ and $x=\frac{c}{b}$. Therefore it is $x(x-a)(x+a)(x-\frac{c}{b})=0$. This may be rewritten as $x(x^2-a^2)(bx-c)=bx^4-cx^3-a^2bx^2+a^2cx=0$.

Vieta's Theorem is generally sufficient to solve problems connected with the roots of quadratic equations. In such questions the roots should never be considered singly, but use should be made of the relations obtained by writing down the sum of the roots, and their product, in terms of the coefficients of the equation.

**Example 5.**
If $\alpha$ and $\beta$ are the roots of $x^2-px+q=0$, find the value of

(a) $\alpha^2+\beta^2$;

(b) $\alpha^3+\beta^3$.

**Solution:**

We have $\alpha+\beta=p$ and $\alpha\beta=q$.

(a) We have $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-2q$.

(b) We have $\alpha^3+\beta^3=(\alpha+\beta)(\alpha^2+\beta^2-\alpha\beta)=p((\alpha+\beta)^2-3\alpha\beta)=p(p^2-3q)$.

**Example 6.**
If $\alpha$ and $\beta$ are the roots of the equation $\ell x^2+mx+n=0$, find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

**Solution:**

We have sum of roots$=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$ and product of roots$=\frac{\alpha}{\beta}\frac{\beta}{\alpha}=1$. By Theorem 2, the required equation is $x^2-(\frac{\alpha^2+\beta^2}{\alpha\beta})x+1=0$, which may be rewritten as $\alpha\beta x^2-(\alpha^2+\beta^2)x+\alpha\beta=0$. As in the last example, $\alpha^2+\beta^2=\frac{m^2-2n\ell}{\ell^2}$ and $\alpha\beta=\frac{n}{\ell}$. So the equation is $\frac{n}{\ell}x^2-\frac{m^2-2n\ell}{\ell^2}x+\frac{n}{\ell}=0$, which may be rewritten as $n\ell x^2-(m^2-2n\ell)x+n\ell=0$.

**Example 7.**
Let $x=\frac{3+5i}{2}$.

(a) Find the value of $2x^3+2x^2-7x+72$.

(b) Show that it will be unaltered if $\frac{3-5i}{2}$ is substituted for $x$ instead.

**Solution:**

Form the quadratic equation whose roots are $x=\frac{3\pm5i}{2}$. Since sum of roots=3 and product of roots$=\frac{17}{2}$, the equation is $2x^2-6x+17=0$.

(a) Perform the following long division.

$$\begin{array}{rrrcrrrr}
 & & & & & & x & +4 \\ \cline{5-8}
2x^2 & -6x & +17 & ) & 2x^3 & +2x^2 & -7x & +72 \\
 & & & & 2x^3 & -6x^2 & +17x & \\ \cline{5-8}
 & & & & & 8x^2 & -24x & +72 \\
 & & & & & 8x^2 & -24x & +68 \\ \cline{6-8}
 & & & & & & & 4
\end{array}$$

When $\frac{3+5i}{2}$ is substituted for $x$, $2x^2-6x+17=0$. Hence

$$2x^3+2x^2-7x+72=(x+4)(2x^2-6x+17)+4=4.$$

(b) Note that $2x^2-6x+17$ is a quadratic *expression* which vanishes for *either* of the values $\frac{3\pm5i}{2}$. Hence when $\frac{3-5i}{2}$ is substituted for $x$, the value of $2x^3+2x^2-7x+72$ will be unaltered.

We now present two minor results about quadratic equations whose roots are related to each other in a special way.

**Theorem 1.**
If the roots of the equation $ax^2 + bx + c = 0$ are equal in magnitude and opposite in signs, then $b = 0$.

**Proof:**

The roots will be equal in magnitude and opposite in sign if their sum is zero. Hence the required condition is $-\frac{b}{a} = 0$, or equivalently $b = 0$.

**Theorem 2.**
If the roots of the equation $ax^2 + bx + c = 0$ are reciprocals of each other, then $c = a$.

**Proof:**

The roots will be reciprocals of each other when their product is 1. Hence the required condition is $\frac{c}{a} = 1$, or equivalently, $c = a$.

Theorem 1 is of frequent occurrence in Analytical Geometry, and Theorem 2 is a particular case of a more general condition applicable to equations of any degree.

**Example 8.**
Find the condition that the roots of $ax^2 + bx + c = 0$ may be

(a) both positive;

(b) opposite in sign, with the negative one numerically greater.

**Solution:**

Let the roots be $\alpha$ and $\beta$. Then $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

(a) If the roots are both positive, $\alpha\beta$ is positive. Therefore $c$ and $a$ have like signs. Also, since $\alpha + \beta$ is positive, $\frac{b}{a}$ is negative. Therefore $b$ and $a$ have unlike signs. In summary, the required condition is that the signs of $a$ and $c$ should be the same, and opposite to the sign of $b$.

(b) If the roots are of opposite signs, $\alpha\beta$ is negative. Therefore $c$ and $a$ have unlike signs. Also since $\alpha + \beta$ has the sign of the numerically greater root, it is negative. Therefore $\frac{b}{a}$ is positive and $b$ and $a$ have like signs. In summary, the required condition is that the signs of $a$ and $b$ should be the same, and opposite to the sign of $c$.

## Quadratic Formula

At this point we revisit the quadratic formula for two reasons. Firstly we wish to extend its validity to the possibility of complex roots. That is, if $b^2 - 4ac$ is negative we shall regard the root as an imaginary quantity and proceed. The second reason is to provide an alternative proof using Vieta's Theorem.

**Proof [Quadratic Formula]:**

Let $\alpha$ and $\beta$ be the roots of $ax^2+bx+c=0$. Then

$$\begin{aligned}\alpha+\beta &= -\frac{b}{a}; && (1)\\ \alpha\beta &= \frac{c}{a}. && (2)\end{aligned}$$

Hence $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(-\frac{b}{a})^2-4(\frac{c}{a})=\frac{b^2-4ac}{a^2}$ by (1) and (2). We may take

$$\alpha-\beta=\frac{\sqrt{b^2-4ac}}{a}. \tag{3}$$

From (1) and (3), we have $\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$. Hence the roots of $ax^2+bx+c=0$ are $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

We can also give a direct verification of Vieta's theorem using the quadratic formula. We have

$$\begin{aligned}\alpha+\beta &= \frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}\\ &= -\frac{2b}{2a}\\ &= -\frac{b}{a}\end{aligned}$$

and

$$\begin{aligned}\alpha\beta &= \frac{-b-\sqrt{b^2-4ac}}{2a}\cdot\frac{-b+\sqrt{b^2-4ac}}{2a}\\ &= \frac{(-b)^2-(b^2-4ac)}{4a^2}\\ &= \frac{4ac}{4a^2}\\ &= \frac{c}{a}.\end{aligned}$$

By writing the equation in the form $x^2+\frac{b}{a}x+\frac{c}{a}=0$, these results may also be expressed as follows. In a quadratic equation where the coefficient of the first term is 1,

(i) the sum of the roots is equal to the coefficient of $x$ with its sign changed;

(ii) the product of the roots is equal to the third term.

**Note**. In any equation the term which does not contain the unknown quantity is frequently called *the constant term*.

We may now extend a result form Chapter I.

**Theorem – Classification of roots**
The expression $b^2 - 4ac$ under the radical is called the *discriminant* of the equation $ax^2 + bx + c = 0$, because the nature of its roots may be determined without solving the equation. Let $\alpha$ and $\beta$ be the roots of $ax^2 + bx + c = 0$. Then,

(i) If $b^2 - 4ac = 0$, then $\alpha = \beta = \frac{b}{2a}$.

(ii) If $b^2 - 4ac$ is a positive square, $\alpha$ and $\beta$ are rational and unequal.

(iii) If $b^2 - 4ac$ is a positive non-square, $\alpha$ and $\beta$ are real and unequal.

(iv) If $b^2 - 4ac$ is negative, $\alpha$ and $\beta$ are conjugate complex numbers.

**Example 1.**
Show that the equation $2x^2 - 6x + 7 = 0$ cannot be satisfied by any real values of $x$.

**Solution:**

Here $a = 2,\ b = -6$ and $c = 7$, so that $b^2 - 4ac = (-6)^2 - 4 \cdot 2 \cdot 7 = -20$. Therefore the roots are not real.

**Example 2.**
If the equation $x^2 + 2(k + 2)x + 9k = 0$ has equal roots, find $k$.

**Solution:**

Using the condition for equal roots, $b^2 - 4ac = 0$ implies $(k + 2)^2 = 9k$. This may be rewritten as $k^2 - 5k + 4 = 0$ or $(k - 4)(k - 1) = 0$. Hence $k = 4$ or $k = 1$.

**Example 3.**
Show that the roots of the equation $x^2 - 2px + p^2 - q^2 + 2qr - r^2 = 0$ are rational.

**Solution:**

The roots will be rational provided $(-2p)^2 - 4(p^2 - q^2 + 2qr - r^2)$ is a square. But this expression reduces to $4(q^2 - 2qr + r^2)$, or $4(q - r)^2$. Hence the roots are rational.

## EXERCISES II

Compute:

1. $(2\sqrt{-3} + 3\sqrt{-2})(4\sqrt{-3} - 5\sqrt{-2})$.

2. $(3\sqrt{-7} - 5\sqrt{-2})(3\sqrt{-7} - 5\sqrt{-2})$.

3. $(e^i + e^{-i})(e^i - e^{-i})$.

4. $(x - \frac{1+\sqrt{-3}}{2})(x - \frac{1-\sqrt{-3}}{2})$.

Express with rational denominator:

5. $\frac{1}{3-\sqrt{-2}}$. 6. $\frac{3\sqrt{-2}+2\sqrt{-5}}{3\sqrt{-2}-2\sqrt{-5}}$. 7. $\frac{3+2i}{2-5i} + \frac{3-2i}{2+5i}$.

8. $\frac{a+xi}{a-xi} - \frac{a-xi}{a+xi}$. 9. $\frac{(x+i)^2}{x-i} - \frac{(x-i)^2}{x+i}$. 10. $\frac{(a+i)^3-(a-i)^3}{(a+i)^2-(a-i)^2}$.

11. Find the value of $(-i)^{4n+3}$, when $n$ is a positive integer.

12. Find the square of $\sqrt{9+40i} + \sqrt{9-40i}$.

Find the square root of:

13. $-5 + 12i$. 14. $-11 - 60i$. 15. $-47 + 8\sqrt{3}i$.

16. $-8i$. 17. $a^2 - 1 + 2ai$. 18. $4ab - 2(a^2 - b^2)i$.

Express in the form $A + Bi$:

19. $\frac{3+5i}{2-3i}$. 20. $\frac{\sqrt{3}-\sqrt{2}i}{2\sqrt{3}-\sqrt{2}i}$. 21. $\frac{1+i}{1-i}$.

22. $\frac{(1+i)^2}{3-i}$.

Form the equations whose roots are:

23. $-\frac{4}{5}, \frac{3}{7}$. 24. $\frac{m}{n}, -\frac{n}{m}$. 25. $\frac{p-q}{p+q}, -\frac{p+q}{p-q}$.

26. $7 \pm 2\sqrt{5}$. 27. $\pm 2\sqrt{3} - 5$. 28. $-p \pm 2\sqrt{2q}$.

29. $-3 \pm 5i$. 30. $-a \pm bi$. 31. $\pm(a - b)i$.

32. $-3,\ \frac{2}{3},\ \frac{1}{2}$. 33. $\frac{a}{2},\ 0,\ -\frac{2}{a}$. 34. $2 \pm \sqrt{3},\ 4$.

35. Prove that the roots of the following equations are real:

(a) $x^2 - 2ax + a^2 - b^2 - c^2 = 0$;

(b) $(a - b + c)x^2 + 4(a - b)x + (a - b - c) = 0$.

36. Prove that the roots of $(x - a)(x - b) = h^2$ are always real.

If $\alpha$ and $\beta$ are the roots of the equation $ax^2 + bx + c = 0$, find the values of

37. $\alpha^4\beta^7 + \alpha^7\beta^4$

38. $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$.

39. $\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^2$

40. $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$.

Find the value of

41. $x^3 + x^2 - x + 22$ when $x = 1 + 2i$.

42. $x^3 - 3x^2 - 8x + 15$ when $x = 3 + i$.

43. $x^3 - ax^2 + 2a^2x + 4a^3$ when $\frac{x}{a} = 1 - \sqrt{-3}$.

If $\alpha$ and $\beta$ are the roots of $x^2 + px + q = 0$, form the equation whose roots are

44. $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

45. $\alpha\beta$ and $\alpha^2 + \beta^2$

46. $\alpha^2 + \beta^2$ and $\alpha^{-2} + \beta^{-2}$.

If $\alpha$ and $\beta$ are the roots of $ax^2 + bx + c = 0$, find the value of

47. $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$;

48. $(a\alpha + b)^{-3} + (a\beta + b)^{-3}$.

49. Find the condition that one root of $ax^2 + bx + c = 0$ shall be $n$ times the other.

50. Discuss the signs of the roots of the equation $px^2 + qx + r = 0$.

51. Form the equation whose roots are the squares of the sum and of the difference of the roots of $2x^2 + 2(m + n)x + m^2 + n^2 = 0$. Hint: See exercise 22.

52. Prove that the modulus of the product of two complex numbers is equal to the product of their moduli. That is: $\text{mod}(x \cdot y) = \text{mod}(x) \cdot \text{mod}(y)$.

# Solutions to Even-numbered Exercises II

2. We have $(3\sqrt{7}i - 5\sqrt{2}i)(3\sqrt{7}i + 5\sqrt{2}i) = i^2((3\sqrt{7})^2 - (5\sqrt{2})^2) = -(63 - 50) = -13$.

4. We have $(x - \frac{1+\sqrt{3}i}{2})(x - \frac{1-\sqrt{3}i}{2}) = x^2 - (\frac{1+\sqrt{3}i}{2} + \frac{1-\sqrt{3}i}{2}) + \frac{1^2-(\sqrt{3}i)^2}{2^2} = x^2 - x + 1$.

6. We have $\frac{3\sqrt{2}i+2\sqrt{5}i}{3\sqrt{2}i-2\sqrt{5}i} = \frac{3\sqrt{2}+2\sqrt{5}}{3\sqrt{2}-2\sqrt{5}} \times \frac{3\sqrt{2}+2\sqrt{5}}{3\sqrt{2}+2\sqrt{5}} = \frac{38+12\sqrt{10}}{-2} = -19 - 6\sqrt{10}$.

8. We have $\frac{a+xi}{a-xi} - \frac{a-xi}{a+xi} = \frac{(a+xi)^2-(a-xi)^2}{a^2-(xi)^2} = \frac{4axi}{a^2+x^2}$.

10. We have $\frac{(a+i)^3-(a-i)^3}{(a+i)^2-(a-i)^2} = \frac{6a^2i-2i}{4ai} = \frac{3a^2-1}{2a}$.

12. We have $(\sqrt{9+40i} + \sqrt{9-40i})^2 = 9 + 40i + 9 - 40i + 2\sqrt{9^2 - (40i)^2} = 18 + 2\sqrt{1681} = 100$.

14. Let $\sqrt{-11-60i} = x + yi$. Then $-11 - 60i = x^2 - y^2 + 2xyi$. Hence $x^2-y^2 = -11$ and $xy = -30$. Now $(x^2+y^2)^2 = (x^2-y^2)^2+4(xy)^2 = 3721$, so that $x^2 + y^2 = 61$. Combined with $x^2 - y^2 = -11$, we have $x^2 = 25$ and $y^2 = 36$. Since $xy = -30$, $x$ and $y$ are of opposite signs. Hence $\sqrt{-11-60i} = \pm(5 - 6i)$.

16. Let $\sqrt{-8i} = x + yi$. Then $-8i = x^2 - y^2 + 2xyi$. Hence $x^2 - y^2 = 0$ and $xy = -4$. From $x^2 - y^2 = 0$, either $y = x$ or $y = -x$. Since $xy = -4$, we must have $y = -x$. Hence $-x^2 = -4$ so that $x = \pm 2$ and $y = \mp 2$. It follows that $\sqrt{-8i} = \pm 2(1 - i)$.

18. Let $\sqrt{4ab - 2(a^2 - b^2)i} = x+yi$. Then $4ab - 2(a^2 - b^2)i = x^2 - y^2 + 2xyi$. Hence $x^2 - y^2 = 4ab$ and $xy = -2(a^2 - b^2)$. Now $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4(xy)^2 = 16a^2b^2 + 4a^4 + 4b^4 - 8a^2b^2$, so that $x^2 + y^2 = 2(a^2 + b^2)$. Combined with $x^2 - y^2 = 4ab$, we have $x^2 = a^2 + b^2 + 2ab$ and $y^2 = a^2 + b^2 - 2ab$. Since $xy = -2(a^2 - b^2)$, $x$ and $y$ are of opposite signs. Hence $\sqrt{4ab - 2(a^2 - b^2)i} = \pm(a + b - (a - b)i)$.

20. We have $\frac{\sqrt{3}-\sqrt{2}i}{2\sqrt{3}-\sqrt{2}i} \times \frac{2\sqrt{3}+\sqrt{2}i}{2\sqrt{3}+\sqrt{2}i} = \frac{6+\sqrt{6}i-2\sqrt{6}i+2}{12+2} = \frac{4}{7} - \frac{\sqrt{6}}{14}i$.

22. We have $\frac{(1+i)^2}{3-i} \times \frac{3+i}{3+i} = \frac{2i(3+i)}{10} = -\frac{1}{5} + \frac{3}{5}i$.

24. The desired equation is $0 = (x - \frac{m}{n})(x - (-\frac{n}{m})) = x^2 - (\frac{m}{n} + (-\frac{n}{m}))x + \frac{m}{n}(-\frac{n}{m}) = x^2 - \frac{m^2-n^2}{mn} - 1$, or $mnx^2 + (n^2 - m^2)x - 1$.

26. The desired equation is

$$\begin{aligned} 0 &= (x - (7 + 2\sqrt{5}))(x - (7 - 2\sqrt{5})) \\ &= x^2 - (7 + 2\sqrt{5} + 7 - 2\sqrt{5})x + (7 + 2\sqrt{5})(7 - 2\sqrt{5}) \\ &= x^2 - 14x + 29. \end{aligned}$$

28. The desired equation is

$$\begin{aligned} 0 &= (x-(-p+2\sqrt{2q}))(x-(-p-2\sqrt{2q})) \\ &= x^2-(-p+2\sqrt{2q}-p-2\sqrt{2q})x+(-p+2\sqrt{2q})(-p-2\sqrt{2q}) \\ &= x^2+2px+p^2-8q. \end{aligned}$$

30. The desired equation is

$$\begin{aligned} 0 &= (x-(-a+bi))(x-(-a-bi)) \\ &= x^2-(-a+bi-a-bi)x+(-a+bi)(-a-bi) \\ &= x^2+2ax+a^2+b^2. \end{aligned}$$

32. The desired equation is

$$\begin{aligned} 0 &= (x-(-3))(x-\frac{2}{3})(x-\frac{1}{2}) \\ &= x^3-(-3+\frac{2}{3}+\frac{1}{2})x^2+(-3(\frac{2}{3})-3(\frac{1}{2})+(\frac{2}{3})(\frac{1}{2})x-(-3)(\frac{2}{3})(\frac{1}{2}) \\ &= x^3+\frac{11}{6}x^2-\frac{19}{6}x+1, \end{aligned}$$

or $6x^3+11x^2-19x+6=0$.

34. The desired equation is

$$\begin{aligned} 0 &= (x-(2+\sqrt{3}))(x-(2-\sqrt{3}))(x-4) \\ &= x^3-(2+\sqrt{3}+2-\sqrt{3}+4)x^2+((2+\sqrt{3})(2-\sqrt{3}) \\ &\quad +4(2+\sqrt{3})+4(2-\sqrt{3}))x \\ &\quad -4(2+\sqrt{3})(2-\sqrt{3}) \\ &= x^3-8x^2-17x-4. \end{aligned}$$

36. We have $x^2-(a+b)x-4(ab-h^2)=0$. So the discriminant is

$$\begin{aligned} \Delta=(a+b)^2-4(ab-h^2) &= a^2+b^2+2ab-4ab+4h^2 \\ &= a^2+b^2-2ab=4h^2 \\ &= (a-b)^2+4h^2 \\ &\geq 0 \end{aligned}$$

38. We have $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$. Hence $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=(-\frac{b}{a})^2-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$, so that $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{(\alpha\beta)^2}=\frac{b^2-2ac}{a^2}\div(\frac{c}{a})^2=\frac{b^2-2ac}{c^2}$.

40. From Problem 16, $\alpha\beta=\frac{c}{a}$ and $\alpha^2+\beta^2=\frac{b^2-2ac}{a^2}$. Since $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2(\alpha\beta)^2$, it is equal to $(\frac{b^2-2ac}{a^2})^2-2(\frac{c}{a})^2=\frac{b^4-4ab^2c+2a^2c^2}{a^4}$. It

follows that

$$\begin{aligned}\left(\frac{\alpha}{\beta}-\frac{\beta}{\alpha}\right)^2 &= \left(\frac{\alpha^2-\beta^2}{\alpha\beta}\right)^2\\ &= \frac{\alpha^4+\beta^4-2(\alpha\beta)^2}{(\alpha\beta)^2}\\ &= \frac{b^4-4ab^2c+2a^2c^2}{a^4}\div\frac{c^2}{a^2}-2\\ &= \frac{b^2(b^2-4ac)}{a^2c^2}.\end{aligned}$$

42. We have $(3+i)^3-3(3+i)^2-8(3+i)+15 = 27+27i-9-i-27-18i+3-24-8i+15 = -15$.

44. We have $\alpha+\beta=-p$ and $\alpha\beta=q$. Then $\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=p^2-2q$. Hence $(\alpha-\beta)^2+(\alpha+\beta)^2=2(\alpha^2+\beta^2)=2(p^2-2q)$ and

$$(\alpha-\beta)^2(\alpha+\beta)^2=(\alpha^2-\beta^2)^2=(\alpha^2+\beta^2)^2-4(\alpha\beta)^2=(p^2-2q)^2-4q^2=p^2(p^2-4q).$$

It follows that the desired equation is $x^2-2(p^2-2q)x+p^2(p^2-4q)=0$.

46. We have $\alpha\beta=q$ and $\alpha^2+\beta^2=p^2-2q$. The desired equation is

$$\begin{aligned}0 &= (x-(\alpha^2+\beta^2))\left(x-\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\right)\\ &= x^2-\left(\alpha^2+\beta^2+\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)x+(\alpha^2+\beta^2)\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\\ &= x^2-(\alpha^2+\beta+2)\frac{(\alpha\beta)^2+1}{(\alpha\beta)^2}x+\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)^2\\ &= x^2-\frac{(p^2-2q)(1+q^2)}{q^2}x+\frac{(p^2-2q)^2}{q^2},\end{aligned}$$

or $q^2x^2-(p^2-2q)(1+q^2)x+(p^2-2q)^2=0$.

48. We have $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$. As in Problem 16, $\alpha^2+\beta^2=\frac{b^2-2ac}{a^2}$. On the other hand, $\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=(-\frac{b}{a})^3-3(-\frac{b}{a})(\frac{c}{a})=\frac{b(3ac-b^2)}{a^3}$ while $(a\alpha+b)(a\beta+b)=a^2\alpha\beta+ab(\alpha+\beta)+b^2=a^2(\frac{c}{a})+ab(-\frac{b}{a})+b^2=ac$.

We have $(a\alpha+b)^3+(a\beta+b)^3=a^3(\alpha^3+\beta^3)+3a^2b(\alpha^2+\beta^2)+3ab^2(\alpha+\beta)+b^3$ which simplifies to $a^3(\frac{b(3ac-b^2)}{a^3})+3a^2b(\frac{b^2-2ac}{a^2})+3ab^2(-\frac{b}{a})+2b^3=b(b^2-3ac)$. Hence $\frac{1}{(a\alpha+b)^3}+\frac{1}{(a\beta+b)^3}=\frac{(a\alpha+b)^3+(a\beta+b)^3}{((a\alpha+b)(a\beta+b))^3}=\frac{b(b^2-3ac)}{a^3c^3}$.

50. Let the roots be $\alpha$ and $\beta$. We assume that they are real, and that $\alpha\geq\beta$. Their signs depend on those of their sum $-\frac{q}{p}$ and their product $\frac{r}{p}$. The following chart summarizes all scenarios.

| Signs | $pr < 0$ | $r = 0$ | $pr > 0$ |
|---|---|---|---|
| $pq < 0$ | $\alpha > 0 > \beta > -\alpha$ | $\alpha > \beta = 0$ | $\alpha > \beta > 0$ |
| $q = 0$ | $\alpha > 0 > \beta = -\alpha$ | $\alpha = \beta = 0$ | Imaginary roots |
| $pq > 0$ | $\alpha > 0 > -\alpha > \beta$ | $0 = \alpha > \beta$ | $0 > \alpha \geq \beta$ |

52. Let the two complex numbers be denoted by $a + bi$ and $c + di$. Then their product is $ac - bd + (ad + bc)i$, which is a complex number whose modulus is

$$\begin{aligned}\sqrt{(ac-bd)^2 + (ad+bc)^2} &= \sqrt{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2} \\ &= \sqrt{(a^2+b^2)(c^2+d^2)} \\ &= \sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}.\end{aligned}$$

### Answers to Odd-numbered Exercises II

1. $6 - 2\sqrt{6}$.

3. $e^{2i} - e^{-2i}$.

5. $\frac{3}{11} + \frac{\sqrt{2}}{11}i$.

7. $-\frac{8}{29}$.

9. $\frac{2(3x^2-1)}{x^2+1}i$.

11. $i$.

13. $\pm(2 + 3i)$.

15. $\pm(1 + 4\sqrt{3}i)$.

17. $\pm(a + i)$.

19. $-\frac{9}{13} + \frac{19}{13}i$.

21. $i$.

23. $35x^2 + 13x - 12 = 0$.

25. $(p^2 - q^2)x^2 + 4pqx - (p^2 - q^2) = 0$.

27. $x^2 + 10x + 13 = 0$.

29. $x^2 + 6x + 34 = 0$.

31. $x^2 + (a - b)^2 = 0$.

33. $2ax^3 + (4 - a^2)x^2 - 2ax = 0$.

37. $\frac{bc^4(3ac-b^2)}{a^7}$.

39. $\frac{b^2}{c^2}$

41. 7.

43. 0.

45. $x^2 - (p^2 - q)x + p^2q - 2q^2 = 0$

47. $\frac{b^2-2ac}{a^2c^2}$.

49. $nb^2 = (1 + n)^2ac$.

51. $x^2 - 4mnx - (m^2 - n^2)^2 = 0$.

# CHAPTER III — MISCELLANEOUS EQUATIONS

In this chapter we propose to consider some miscellaneous equations; it will be seen that many equations can be solved by the ordinary rules for quadratic equations, but others require some special artifice for their solution.

**Example 1.**
Solve $8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63$.

**Solution:**

Let $y = x^{\frac{3}{2n}}$. Then $8y - \frac{8}{y} = 63$. Hence $0 = 8y^2 - 63y - 8 = (8y+1)(y-8)$, so that $y = -\frac{1}{8}$ or 8. When $y = -\frac{1}{8}$, $x = (-\frac{1}{2^3})^{\frac{2n}{3}} = \frac{1}{2^{2n}}$. When $y = 8$, $x = (2^3)^{\frac{2n}{3}} = 2^{2n}$.

**Example 2.**
Solve $2\sqrt{\frac{x}{a}} + 3\sqrt{\frac{a}{x}} = \frac{b}{a} + \frac{6a}{b}$.

**Solution:**

Let $y = \sqrt{\frac{x}{a}}$. Then $2y + \frac{3}{y} = \frac{b}{a} + \frac{6a}{b}$. Hence $0 = 2aby^2 - 6a^2y - b^2y + 3ab = (2ay-b)(by-3a)$, so that $y = \frac{b}{2a}$ or $\frac{3a}{b}$. When $y = \frac{b}{2a}$, $\frac{x}{a} = \frac{b^2}{4a^2}$ so that $x = \frac{b^2}{4a}$. When $y = \frac{3a}{b}$, $\frac{x}{a} = \frac{9b^2}{a^2}$ so that $x = \frac{9a^3}{b^2}$.

**Example 3.**
Solve $(x-5)(x-7)(x+6)(x+4) = 504$.

**Solution:**

We have $(x-5)(x+4) = x^2-x-20$ while $(x-7)(x+6) = x^2-x-42$. Let $y = x^2-x-20$. Then $y(y-22) = 504$. Hence $0 = y^2-22y-504 = (y+14)(y-36)$, so that $y = -14$ or 36. When $y = -14$, $0 = x^2 - x - 6 = (x+2)(x-3)$, so that $x = -2$ or 3. When $y = 36$, $0 = x^2 - x - 56 = (x+7)(x-8)$, so that $x = -7$ or 8.

Any equation which can be thrown into the form $ax^2+bx+c+p\sqrt{ax^2+bx+c} = q$ may be solved as follows. Putting $y = \sqrt{ax^2+bx+c}$, we obtain $y^2+py-q = 0$. Let $\alpha$ and $\beta$ be the roots of this equation. Then $\sqrt{ax^2+bx+c} = \alpha$ or $\sqrt{ax^2+bx+c} = \beta$. From these equations we shall obtain *four* values of $x$.

When no sign is prefixed to a radical, it is usually understood that it is to be taken as positive. Hence, if $\alpha$ and $\beta$ are both positive, all the four values of $x$ satisfy the original equation. If however $\alpha$ or $\beta$ is negative, the roots found from the resulting quadratic will satisfy the equation $ax^2+bx+c-p\sqrt{ax^2+bx+c} = q$ but not the original equation. Thus they are to be discarded. These false roots are called *extraneous roots*.

**Example 4.**
Solve $x^2 - 5x + 2\sqrt{x^2-5x+3} = 12$.

**Solution:**

Let $y = \sqrt{x^2 - 5x + 3}$. Then $y^2 - 3 + 2y = 12$. Hence $0 = y^2 + 2y - 15 = (y + 5)(y - 3)$, so that $y = -5$ or 3. However, $\sqrt{x^2 - 5x + 3} \geq 0$, so that we cannot have $y = -5$. When $y = 3$, $0 = x^2 - 5x - 6 = (x + 1)(x - 6)$, so that $x = -1$ or 6.

Before clearing an equation of radicals, it is advisable to examine whether any common factor can be removed by division.

**Example 5.**
Solve $\sqrt{x^2 - 7ax + 10a^2} - \sqrt{x^2 + ax - 6a^2} = x - 2a$.

**Solution:**

Factorization yields $\sqrt{(x - 2a)(x - 5a)} - \sqrt{(x - 2a)(x + 3a)} = x - 2a$. Thus one root is $x = 2a$. When $x \neq 2a$, we may divide throughout by $\sqrt{x - 2a}$ and obtain $\sqrt{x - 5a} - \sqrt{x + 3a} = \sqrt{x - 2a}$. Squaring both sides yields $x - 5a + x + 3a - 2\sqrt{(x - 5a)(x + 3a)} = x - 2a$, which may be rewritten as $x = 2\sqrt{x^2 - 2ax - 15a^2}$. Squaring both sides again yields $0 = 3x^2 - 8ax - 60a^2 = (3x + 10a)(x - 6a)$, so that $x = -\frac{10a}{3}$ or $6a$. However, $x = 6a$ is an extraneous root which does not satisfy the given equation. Hence the only other root is $x = -\frac{10a}{3}$.

In this example we again encounter an extraneous root. This time the extraneous root arises when we square both sides. In general it is necessary to check the validity of each root by substitution into the original expression.

The following artifice is sometimes useful.

**Example 6.**
Solve $\sqrt{x^2 - 4x + 34} + \sqrt{3x^2 - 4x - 11} = 9$.

**Solution:**

Note that $(3x^2 - 4x + 34) - (3x^2 - 4x - 11) = 45$. Dividing this by the given equation, making use of the factorization of the difference of two squares, we have $\sqrt{3x^2 - 4x + 34} - \sqrt{3x^2 - 4x - 11} = 5$. Now this equation is only true for the same values of $x$ which satisfies the given equation. Adding the two equations, we have $2\sqrt{3x^2 - 4x + 34} = 14$. Hence $0 = 3x^2 - 4x - 15 = (3x + 5)(x - 3)$, so that $x = -\frac{5}{3}$ or 3.

The solution of an equation of the form $ax^4 \pm bx^3 \pm cx^2 \pm bx + a = 0$, in which the coefficients of terms equidistant from the beginning and end are equal, can be made to depend on the solution of a quadratic. Equations of this type are known as reciprocal equations, and are so named because they are not altered when $x$ is changed into its reciprocal $\frac{1}{x}$. We shall revisit the notion of reciprocal equations in a later chapter.

**Example 7.**
Solve $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.

**Solution:**

First, note that $x = 0$ is not a root. Dividing throughout by $x^2$, $12(x^2 + \frac{1}{x^2}) - 56(x + \frac{1}{x}) + 89 = 0$. Let $y = x + \frac{1}{x}$. Then $12(y^2 - 2) - 56y + 89 = 0$. Hence $0 = 12y - 56y - 65 = (6y - 13)(2y - 5)$, so that $y = \frac{13}{6}$ or $\frac{5}{2}$. When $y = \frac{13}{6} = x + \frac{1}{x}$, $0 = 6x^2 - 13x + 6 = (3x - 2)(2x - 3)$, so that $x = \frac{2}{3}$ or $\frac{3}{2}$. When $y = \frac{5}{2} = x + \frac{1}{x}$, $0 = 2x^2 - 5x + 2 = (2x - 1)(x - 2)$, so that $x = \frac{1}{2}$ or 2.

The following equation, though *not* reciprocal, may be solved in a similar manner.

**Example 8.**
Solve $6x^4 - 25x^3 + 12x^2 + 25x + 6 = 0$.

**Solution:**

First, note that $x = 0$ is not a root. Dividing throughout by $x^2$, $6(x^2 + \frac{1}{x^2}) - 25(x - \frac{1}{x}) + 12 = 0$. Let $y = x - \frac{1}{x}$. Then $6(y^2 + 2) - 25y + 12 = 0$. Hence $0 = 6y^2 - 25y + 24 = (2y - 3)(3y - 8)$, so that $y = \frac{3}{2}$ or $\frac{8}{3}$. When $y = \frac{3}{2} = x - \frac{1}{x}$, $0 = 2x^2 - 3x - 2 = (2x + 1)(x - 2)$, so that $x = -\frac{1}{2}$ or 2. When $y = \frac{8}{3} = x - \frac{1}{x}$, $0 = 3x^2 - 8x - 3 = (3x + 1)(x - 3)$, so that $x = -\frac{1}{3}$ or 3.

When one root of a quadratic equation is obvious by inspection, the other root may often be readily obtained by making use of the properties of the roots of quadratic equations.

**Example 9.**
Solve $(1 - a^2)(x + a) - 2a(1 - x^2) = 0$.

**Solution:**

This is a quadratic, one of whose roots is clearly $a$. Also, since the equation may be rewritten as $2ax^2 + (1 - a^2)x - a(1 + a^2) = 0$, the product of the roots is $-\frac{1+a^2}{2}$. Hence the other root is $-\frac{1+a^2}{2a}$.

## EXERCISES III

Solve the following equations:

1. $x^{-2} - 2x^{-1} = 8$.
2. $9 + x^{-4} = 10x^{-2}$.
3. $2\sqrt{x} + \frac{2}{\sqrt{x}} = 5$.
4. $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}$.
5. $x^{\frac{2}{n}} + 6 = 5x^{\frac{1}{n}}$.
6. $3x^{\frac{1}{2n}} - x^{\frac{1}{n}} - 2 = 0$.
7. $5\sqrt{\frac{3}{x}} + 7\sqrt{\frac{x}{3}} = 22\frac{2}{3}$.
8. $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$.
9. $6\sqrt{x} = 5x^{-\frac{1}{2}} - 13$.
10. $1 + 8x^{\frac{6}{5}} + 9x^{\frac{3}{5}} = 0$.

11. $3^{2x} + 9 = 10(3^x)$.

12. $5(5^x + 5^{-x}) = 26$.

13. $2^{2x+8} + 1 = 32(2^x)$.

14. $2^{2x+3} - 57 = 65(2^x - 1)$.

15. $\sqrt{2^x} + \frac{1}{\sqrt{2^x}} = 2$.

16. $\frac{3}{\sqrt{2x}} - \frac{\sqrt{2x}}{5} = \frac{59}{10}$.

17. $(x-7)(x-3)(x+5)(x+1) = 1680$.

18. $(x+9)(x-3)(x-7)(x+5) = 385$.

19. $x(2x+1)(x-2)(2x-3) = 63$.

20. $(2x-7)(x^2-9)(2x+5) = 91$.

21. $x^2 + 2\sqrt{x^2+6x} = 24 - 6x$.

22. $3x^2 - 4x + \sqrt{3x^2-4x-6} = 18$.

23. $3x^2 - 7 + 3\sqrt{3x^2-16x+21} = 16x$.

24. $8 + 9\sqrt{(3x-1)(x-2)} = 3x^2 - 7x$.

25. $\frac{3x-2}{2} + \sqrt{2x^2-5x+3} = \frac{(x+1)^2}{3}$.

26. $7x - \frac{\sqrt{3x^2-8x+1}}{x} = (\frac{8}{\sqrt{x}} + \sqrt{x})^2$.

27. $\sqrt{4x^2-7x-15} - \sqrt{x^2-3x} = \sqrt{x^2-9}$.

28. $\sqrt{2x^2-9x+4} + 3\sqrt{2x-1} = \sqrt{2x^2+21x-11}$.

29. $\sqrt{2x^2+5x-7} + \sqrt{3(x^2-7x+6)} - \sqrt{7x^2-6x-1} = 0$.

30. $\sqrt{a^2+2ax-3x^2} - \sqrt{a^2+ax-6x^2} = \sqrt{2a^2+3ax-9x^2}$.

31. $\sqrt{2x^2+5x-2} - 2\sqrt{2x^2+5x-9} = 1$.

32. $\sqrt{3x^2-2x+9} + \sqrt{3x^2-2x-4} = 13$.

33. $\sqrt{2x^2-7x+1} - \sqrt{2x^2-9x+4} = 1$.

34. $\sqrt{3x^2-7x-30} - \sqrt{2x^2-7x-5} = x - 5$.

35. $x^4 + x^3 - 4x^2 + x + 1 = 0$.

36. $x^4 + \frac{8}{9}x^2 + 1 = 3x^3 + 3x$.

37. $x^4 + 1 - 3(x^3 + x) = 2x^3$.

38. $10(x^4+1) - 63x(x^2-1) + 52x^2 = 0$.

39. $\frac{x+\sqrt{12a-x}}{x-\sqrt{12a-x}} = \frac{\sqrt{a}+1}{\sqrt{a}-1}$.

40. $\frac{a+2x+\sqrt{a^2-4x^2}}{a+2x-\sqrt{a^2-4x^2}} = \frac{5x}{a}$.

41. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} - \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 8x\sqrt{x^2-3x+2}$.

42. $\sqrt{x^2+x} + \frac{\sqrt{x-1}}{\sqrt{x^3-x}} = \frac{5}{2}$.

43. $\frac{x^3+1}{x^2-1} = x + \sqrt{\frac{6}{x}}$.

44. $\frac{2^{x^2}}{2^{2x}} = \frac{8}{1}$.

45. $a^{2x}(a^2+1) = (a^{3x}+a^x)a$.

46. $\frac{8\sqrt{x-5}}{3x-7} = \frac{\sqrt{3x-7}}{x-5}$.

47. $\frac{18(7x-3)}{2x+1} = \frac{250\sqrt{2x+1}}{3\sqrt{7x-3}}$.

48. $(a+x)^{\frac{2}{3}} + 4(a-x)^{\frac{2}{3}} = 5(a^2-x^2)^{\frac{1}{3}}$.

49. $\sqrt{x^2+ax-1} - \sqrt{x^2+bx-1} = \sqrt{a} - \sqrt{b}$.

50. $\frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}} + \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = 98$.

51. $x^4 - 2x^3 + x = 380$.

52. $27x^3 + 21x + 8 = 0$.

## Solutions to Even-numbered Exercises III

2. Let $y = x^{-2}$. Then $9+y^2 = 10y$. Hence $0 = y^2-10y+9 = (y-1)(y-9)$, so that $y = 1$ or 9. When $y = 1$, $x = \pm 1$. When $y = 9$, $x = \pm\frac{1}{3}$.

4. First, note that $x \neq 0$. Multiplying throughout by $x^{\frac{1}{4}}$, we have $6x = 7\sqrt{x} - 2$. Let $y = \sqrt{x}$. Then $6y^2 = 7y - 2$. Hence $0 = 6y^2 - 7y + 2 = (2y-1)(3y-2)$, so that $y = \frac{1}{2}$ or $\frac{2}{3}$. When $y = \frac{1}{2}$, $x = \frac{1}{4}$. When $y = \frac{2}{3}$, $x = \frac{4}{9}$.

6. Let $y = x^{\frac{1}{2n}}$. Then $3y-y^2-2 = 0$. Hence $0 = y^2-3y+2 = (y-1)(y-2)$, so that $y = 1$ or 2. When $y = 1$, $x = 1$. When $y = 2$, $x = 2^{2n}$.

8. Let $y = \sqrt{\frac{x}{1-x}}$. Then $y+\frac{1}{y} = \frac{13}{6}$. Hence $0 = 6y^2-13y+6 = (2y-3)(3y-2)$, so that $y = \frac{2}{3}$ or $\frac{3}{2}$. When $y = \frac{2}{3}$, $\frac{x}{1-x} = \frac{4}{9}$. Hence $9x = 4 - 4x$ so that $x = \frac{4}{13}$. When $y = \frac{3}{2}$, $\frac{x}{1-x} = \frac{9}{4}$. Hence $4x = 9 - 9x$ so that $x = \frac{9}{13}$.

10. Let $y = x^{\frac{3}{5}}$. Then $1 + 8y^2 + 9y = 0$. Hence $0 = 8y^2 + 9y + 1 = (8y+1)(y+1)$, so that $y = -\frac{1}{8}$ or $-1$. When $y = -\frac{1}{8}$, $x = -\frac{1}{32}$. When $y = -1$, $x = -1$.

12. Let $y = 5^x$. Then $5y+\frac{5}{y} = 26$. Hence $0 = 5y^2-26y+5 = (5y-1)(y-5)$, so that $y = \frac{1}{5}$ or 5. When $y = \frac{1}{5}$, $x = -1$. When $y = 5$, $x = 1$.

14. Let $y = 2^x$. Then $8y^2 - 57 = 65(y-1)$. Hence $0 = 8y^2 - 65y + 8 = (8y-1)(y-8)$, so that $y = \frac{1}{8}$ or 8. When $y = \frac{1}{8}$, $x = -3$. When $y = 8$, $x = 3$.

16. Let $y = \sqrt{2x}$. Then $\frac{3}{y} - \frac{y}{5} = \frac{59}{10}$. Hence $0 = 2y^2 + 59y - 30 = (y + 30)(2y - 1)$, so that $y = -30$ or $\frac{1}{2}$. However, $\sqrt{2x} \geq 0$, so that $y \neq -30$. When $y = \frac{1}{2}$, $x = \frac{1}{8}$.

18. We have $(x+9)(x-7) = x^2+2x-63$ while $(x-3)(x+5) = x^2+2x-15$. Let $y = x^2+2x-15$. Then $y(y-48) = 385$. Hence $0 = y^2-48y+385 = (y+7)(y-55)$, so that $y = -7$ or $55$. When $y = -7$, $0 = x^2+2x-8 = (x+4)(x-2)$, so that $x = -4$ or $2$. When $y = 55$, $x^2+2x-70 = 0$ so that $x = \frac{-2\pm\sqrt{4+280}}{2} = -1 \pm \sqrt{71}$.

20. We have $(2x-7)(x+3) = 2x^2-x-21$ while $(2x+5)(x-3) = 2x^2-x-15$. Let $y = 2x^2-x-15$. Then $y(y-6) = 91$. Hence $0 = y^2-6y-91 = (y+7)(y-13)$, so that $y = -7$ or $13$. When $y = -7$, $2x^2-x-8 = 0$, so that $x = \frac{1\pm\sqrt{1+64}}{4} = \frac{1\pm\sqrt{65}}{4}$. When $y = 13$, $0 - 2x^2 - x - 28 = (2x+7)(x-4)$, so that $x = -\frac{7}{2}$ or $4$.

22. Let $y = \sqrt{3x^2 - 4x - 6}$. Then $y^2 + 6 + y = 18$. Hence $0 = y^2 + y - 12 = (y+4)(y-3)$, so that $y = -4$ or $3$. However, $\sqrt{3x^2 - 4x - 6} \geq 0$, so that we cannot have $y = -4$. When $y = 3$, $0 = 3x^2-4x-15 = (3x+5)(x-3)$, so that $x = -\frac{5}{3}$ or $3$.

24. Let $y = \sqrt{3x^2 - 7x + 2}$. Then $8+9y = y^2-2$. Hence $0 = y^2-9y-10 = (y+1)(y-10)$, so that $y = -1$ or $10$. However, $\sqrt{3x^2 - 7x + 2} \geq 0$, so that we cannot have $y = -1$. When $y = 10$, $0 = 3x^2 - 7x - 98 = (3x+14)(x-7)$, so that $x = -\frac{14}{3}$ or $7$.

26. First, note that $x \neq 0$. Multiplying throughout by $x$, we have $7x^2 + \sqrt{3x^2 - 8x + 1} = 64 + 16x + x^2$, which may be rewritten as $6x^2 - 16x + \sqrt{3x^2 - 8x + 1} = 64$. Let $y = \sqrt{3x^2 - 8x + 1}$. Then $2y^2 - 2 + y = 64$. Hence $0 = 2y^2+y-66 = (y+6)(2y-11)$, so that $y = -6$ or $\frac{11}{2}$. However, $\sqrt{3x^2 - 8x + 1} \geq 0$, so that $y \neq -6$. When $y = \frac{11}{2}$, $12x^2-32x-117 = 0$, so that $x = \frac{32\pm\sqrt{1024+5616}}{24} = \frac{8\pm\sqrt{415}}{6}$.

28. Factorization yields $\sqrt{(2x-1)(x-4)} + 3\sqrt{2x-1} = \sqrt{(2x-1)(x+11)}$. Thus one root is $x = \frac{1}{2}$. When $x \neq \frac{1}{2}$, we may divide throughout by $\sqrt{2x-1}$ and obtain $\sqrt{x-4} + 3 = \sqrt{x+11}$. Squaring both sides yields $x - 4 + 9 + 6\sqrt{x-4} = x + 11$, which may be rewritten as $\sqrt{x-4} = 1$. Hence $x - 4 = 1$ and $x = 5$ is the other root.

30. Factorization yields $\sqrt{(a+3x)(a-x)} - \sqrt{(a+3x)(a-2x)} = \sqrt{(a+3x)(2a-3x)}$. Thus one root is $x = -\frac{a}{3}$. When $x \neq -\frac{a}{3}$, we may divide throughout by $a + 3x$. The resulting equation is $\sqrt{a-x} - \sqrt{a-2x} = \sqrt{2a-3x}$. Squaring yields $a - x + a - 2x - 2\sqrt{(a-x)(a-2x)} = 2a - 3x$, which may be rewritten as $\sqrt{(a-x)(a-2x)} = 0$. Hence $(a-x)(a-2x) = 0$, so that

$x = a$ and $x = \frac{a}{2}$. However, $x = a$ is an extraneous root which does not satisfy the given equation. Hence the only other root is $x = \frac{a}{2}$.

32. Note that $(3x^2 - 2x + 9) - (3x^2 - 2x - 4) = 13$. Dividing this by the given equation, we have $\sqrt{3x^2 - 2x + 9} - \sqrt{3x^2 - 2x - 4} = 1$. Adding this to the given equation, we have $2\sqrt{3x^2 - 2x + 9} = 14$. Hence $0 = 3x^2 - 2x - 40 = (3x + 10)(x - 4)$, so that $x = -\frac{10}{3}$ or 4.

34. Squaring both sides yields $3x^2 - 7x - 30 = 2x^2 - 7x - 5 + x^2 - 10x + 25 + 2(x - 5)\sqrt{2x^2 - 7x - 5}$, which may be rewritten as $5(x - 5) = (x - 5)\sqrt{2x^2 - 7x - 5}$. Thus one root is $x = 5$. When $x \neq 5$, we may divide throughout by $x - 5$ and obtain $5 = \sqrt{2x^2 - 7x - 5}$. Hence $0 = 2x^2 - 7x - 30 = (2x + 5)(x - 6)$ and $x = -\frac{5}{2}$ and $x = 6$ are the other roots.

36. First, note that $x = 0$ is not a root. Dividing throughout by $x^2$, we have $x^2 + \frac{8}{9} + \frac{1}{x^2} = 3x + \frac{3}{x}$. Let $y = x + \frac{1}{x}$. Then $y^2 - 2 + \frac{8}{9} = 3y$. Hence $0 = 9y^2 - 27y - 10 = (3y + 1)(3y - 10)$, so that $y = -\frac{1}{3}$ or $\frac{10}{3}$. When $y = -\frac{1}{3} = x + \frac{1}{x}$, $3x^2 + x + 3 = 0$ so that $x = \frac{-1 \pm \sqrt{1-36}}{2} = \frac{-1 \pm \sqrt{35}i}{2}$. When $y = \frac{10}{3} = x + \frac{1}{x}$, $0 = 3x^2 - 10x + 3 = (3x - 1)(x - 3)$, so that $x = \frac{1}{3}$ or 3.

38. First, note that $x = 0$ is not a root. Dividing throughout by $x^2$, $10(x^2 + \frac{1}{x^2}) - 63(x - \frac{1}{x}) + 52 = 0$. Let $y = x - \frac{1}{x}$. Then $10(y^2 - 2) - 63y + 52 = 0$. Hence $0 = 10y^2 - 63y + 72 = (2y - 3)(5y - 24)$, so that $y = \frac{3}{2}$ or $\frac{24}{5}$. When $y = \frac{3}{2} = x - \frac{1}{x}$, $0 = 2x^2 - 3x - 2 = (2x + 1)(x - 2)$, so that $x = -\frac{1}{2}$ or 2. When $y = \frac{24}{5} = x - \frac{1}{x}$, $0 = 5x^2 - 24x - 5 = (5x + 1)(x - 5)$ so that $x = -\frac{1}{5}$ or 5.

40. Rationalizing the denominator, we have $\frac{a^2+4ax+4x^2+a^2-4x^2+2(a+2x)\sqrt{a^2-4x^2}}{a^2+4ax+4x^2-a^2+4x^2} = \frac{5x}{a}$, which may be rewritten as $\frac{(a+2x)(a+\sqrt{a^2-4x^2})}{2x(a+2x)} = \frac{5x}{a}$. Note that $x(a + 2x) \neq 0$. Hence $\frac{a+\sqrt{a^2-4x^2}}{2x} = \frac{5x}{a}$, which may be rewritten as $10x^2 - a^2 = a\sqrt{a^2 - 4x^2}$. Squaring yields $100x^2 - 20a^2x^2 + a^4 = a^4 - 4a^2x^2$. Since $x \neq 0$, we have $25x^2 = 4a^2$ so that $x = \pm\frac{2a}{5}$.

42. First, note that $x \neq 1$. Hence the equation may be rewritten as $\sqrt{x^2 + x} + \frac{1}{\sqrt{x^2+x}} = \frac{5}{2}$. Let $y = \sqrt{x^2 + x}$. Then $y + \frac{1}{y} = \frac{5}{2}$. Hence $0 = 2y^2 - 5y + 2 = (2y - 1)(y - 2)$, so that $y = \frac{1}{2}$ or 2. When $y = \frac{1}{2}$, $x^2 + x = \frac{1}{4}$. Hence $4x^2 + 4x - 1 = 0$, so that $x = \frac{-4 \pm \sqrt{16+16}}{8} = \frac{-1 \pm \sqrt{2}}{2}$. When $y = 2$, $x^2 + x = 4$. Hence $x^2 + x - 4 = 0$, so that $x = \frac{-1 \pm \sqrt{1+16}}{2} = \frac{-1 \pm \sqrt{17}}{2}$.

44. The given equation may be rewritten as $2^{x^2} == 2^{2x+3}$. Hence $0 = x^2 - 2x - 3 = (x + 1)(x - 3)$, so that $x = -1$ or 3.

46. The given equation may be rewritten as $8(x - 5)^{\frac{3}{2}} = (3x - 7)^{\frac{3}{2}}$. Squaring both sides yields $64(x - 5)^3 = (3x - 7)^3$ and taking cube roots yields $4(x - 5) = 3x - 7$, so that $x = 13$.

48. We have $0 = (a+x)^{\frac{2}{3}} - 5(a+x)^{\frac{1}{3}}(a-x)^{\frac{1}{3}} + 4(a-x)^{\frac{2}{3}} = ((a+x)^{\frac{1}{3}} - (a-x)^{\frac{1}{3}})((a+x)^{\frac{1}{3}} - 4(a-x)^{\frac{1}{3}})$. If the first factor is 0, then $a+x = a-x$ so that $x = 0$. If the second factor is 0, then $a+x = 64(a-x)$, so that $x = \frac{63a}{65}$.

50. Rationalizing the denominators, we have $x^2 + x^2 - 1 + 2x\sqrt{x^2-1} + x^2 + x^2 - 1 - 2x\sqrt{x^2-1} = 98$, which may be rewritten as $x^2 = 25$. Hence $x = \pm 5$.

52. By inspection, one of the roots is $x = -\frac{1}{3}$. When $x \neq -\frac{1}{3}$, we may divide throughout by $x+3$ and obtain $9x^2 - 3x + 8 = 0$, so that the other roots are $x = \frac{3\pm\sqrt{9-288}}{18} = \frac{1\pm\sqrt{31}i}{6}$.

## Answers to Odd-numbered Exercises III

1. $-\frac{1}{2}, \frac{1}{4}$.
3. $\frac{1}{4}, 4$.
5. $2^n, 3^n$,
7. $\frac{25}{147}, 27$.
9. $\frac{1}{9}, \frac{25}{4}$.
11. 0,2.
13. $-4$.
15. 0.
17. $-7, 9, 1 \pm 2\sqrt{6}i$.
19. $-\frac{3}{2}, 3, \frac{3\pm\sqrt{47}i}{4}$.
21. $-8, 2$.
23. $\frac{1}{3}, 5$.
25. $\frac{1}{2}, 2, \frac{5\pm\sqrt{201}}{4}$.
27. 1,3.
29. $-\frac{18}{5}, 1, 9$.
31. $-\frac{9}{2}, 2$.
33. 0,5.
35. $1, \frac{-3\pm\sqrt{5}}{2}$.
37. $2 \pm \sqrt{3}, \frac{-1\pm\sqrt{3}i}{2}$.
39. $-4a, 3a$.
41. 0,1,3.
43. $\frac{3}{2}$.
45. $\pm 1$.
47. 4.
49. $1, \frac{(\sqrt{a}-\sqrt{b})^2+4}{(\sqrt{a}+\sqrt{b})^2-4}$.
51. $-4, 5, \frac{1\pm 5\sqrt{3}i}{2}$.

# CHAPTER IV — Equations of Higher Order

1. The Fundamental Theorem, and Others
2. Transformations

## The Fundamantal Theorem, and Others

Let $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$ be a polynomial in $x$ of degree $n$. Then $f(x) = 0$ is the general polynomial equation of degree $n$. Dividing throughout if necessary by $p_0 \neq 0$, we see that without any loss of generality we may take $p_0 = 1$. Unless otherwise stated, the coefficients $p_1, p_2, \ldots, p_n$ will always be supposed rational. Any value of $x$ which makes $f(x)$ vanish is called a root of the equation $f(x) = 0$.

**Remainder Theorem.**
When a polynomial $f(x)$ in $x$ is divided by $x - \alpha$, the remainder is $f(\alpha)$.

**Proof:**

Since $f(x)$ is divided by a linear polynomial $x - \alpha$, the remainder does not involve $x$. Let $Q(x)$ be the quotient and $R$ the remainder. Then $f(x) = (x-\alpha)Q(x)+R$. Since $R$ does not involve $x$, it will remain unaltered whatever value we give to $x$. Put $x = \alpha$. Then $f(\alpha) = (\alpha - \alpha)Q(\alpha) + R$. Since $Q(x)$ is finite for finite values of $x$, $R = f(\alpha)$.

**Factor Theorem.**
If $f(\alpha) = 0$, then $f(x)$ is divisible by $x - \alpha$, and conversely.

**Proof:**

By the Remainder Theorem, both conditions are equivalent to $R = 0$.

**Example 1.**
Find the equation connecting $a$ and $b$ in order that $2x^4 - 7x^3 + ax + b$ may be divisible by $x - 3$.

**Solution**

Let $f(x) = 2x^4 - 7x^3 + ax + b$. Then $f(3) = 162 - 189 + 3a + b = 3a + b - 27$. Hence the desired equation is $3a + b = 27$.

**Fundamental Theorem of Algebra.**
Every polynomial equation with complex coefficients has a root, real or imaginary.

This result was by Gauss for his doctoral thesis and the proof is beyond the scope of the present work. From this, it can be deduced that a polynomial equation of degree $n$ has exactly $n$ roots. Denote the given equation by $f(x) = 0$. Then $f(x) = 0$ bas a root $\alpha_1$ and $f(x)$ is divisible by $x - \alpha_1$, so that $f(x) = (x - \alpha_1)f_1(x)$ for some polynomial $f_1(x)$ of degree $n - 1$. Again, the equation $f_1(x) = 0$ has a root $\alpha_2$. Then $f_1(x)$ is divisible by $x - \alpha_2$, so that $f_1(x) = (x - \alpha_2)f_2(x)$ for some polynomial $f_2(x)$ of degree $n - 2$. Proceeding in this way, we obtain

$$f(x) = (x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n).$$

Hence the equation $f(x) = 0$ has $n$ roots, since $f(x)$ vanishes when $x$ has any of the values $\alpha_1, \alpha_2, \dots, \alpha_n$.

Also the equation cannot have more than $n$ roots; for if $x$ has any value different from any of the quantities $\alpha_1, \alpha_2, \dots, \alpha_n$, all the factors on the right are different from zero, and therefore $f(x)$ cannot vanish for that value of $x$.

In the above investigation some of the quantities $\alpha_1, \alpha_2, \dots, \alpha_n$ may be equal. In this case, however, we shall suppose that the equation still has $n$ roots, although these are not all different.

### Vieta's Theorem for n$^{\text{th}}$ degree polynomials

We now investigate the relations between the roots and the coefficients in any equation. Let us denote the equation by $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$, and the roots by $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$. Then we have identically

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_1x + p_0 = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)\cdots(x - \alpha_n).$$

Equating coefficients of like powers of $x$ in this identity,

$$\begin{aligned} -p_1 &= \text{sum of the roots;} \\ p_2 &= \text{sum of the products of the roots taken two at a time;} \\ -p_3 &= \text{sum of the products of the roots taken three at a time;} \\ \cdots &= \cdots; \\ (-1)^n p_n &= \text{product of the roots.} \end{aligned}$$

**Example 2.**

If $\alpha, \beta$ and $\gamma$ are the roots of the equation $x^3 + px^2 + qx + r = 0$, form the equation whose roots are $\alpha^2, \beta^2$ and $\gamma^2$.

**Solution:**

The required equation is $(y - \alpha^2)(y - \beta^2)(y - \gamma^2) = 0$, or $(x^2 - \alpha^2)(x^2 - \beta^2)(x^2 - \gamma^2) = 0$ if $y = x^2$. In other words, $(x - \alpha)(x - \beta)(x - \gamma)(x + \alpha)(x + \beta)(x + \gamma) = 0$. However, $(x - \alpha)(x - \beta)(x - \gamma) = x^3 + px^2 + qx + r$. Hence $(x + \alpha)(x + \beta)(z + \gamma) = x^3 - px^2 + qx - r$. Thus the required equation is

$$\begin{aligned} 0 &= (x^3 + px2 + qx + r)(x^2 - px^2 + qx - r) \\ &= (x^3 + qx)^2 - (px^2 + r)^2 \\ &= x^6 + (2q - p^2)x^4 + (q^2 - 2pr)x^2 - r^2. \end{aligned}$$

If we replace $x^2$ by $y$, we obtain $y^3 + (2q - p^2)y^2 + (q^2 - 2pr)y - r^2 = 0$.

## A Caution for Vieta's Theorem

The student might suppose that the relations established so far would enable him or her to solve any proposed equation; for the number of the relations is equal to the number of the roots. A little reflection will show that this is not the case. Suppose we eliminate any $n-1$ of the roots $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ in order to obtain an equation to determine the remaining one. Since these roots are involved symmetrically in each of the equations, it is clear that we shall always obtain an equation having the same coefficients; this equation is therefore the original equation with some one of the roots $\alpha_k$ substituted for $x$.

Let us take for example the equation $x^3+px^2+qx+r=0$. Let $\alpha, \beta$ and $\gamma$ be the roots. Then

$$\begin{aligned} \alpha+\beta+\gamma &= -p, \\ \alpha\beta+\beta\gamma+\gamma\alpha &= q, \\ \alpha\beta\gamma &= -r. \end{aligned}$$

Multiply these equations by $\alpha^2, -\alpha$ and 1 respectively and add, we obtain $\alpha^3 = -p\alpha^2 - q\alpha - r$, that is, $\alpha^3 + p\alpha^2 + q\alpha + r = 0$, which is the original equation with $\alpha$ in the place of $x$.

The above process of elimination is quite general, and is applicable to equations of any degree.

If two or more of the roots of an equation are connected by an assigned relation, the properties proved earlier will sometimes enable us to obtain the complete solution.

**Example 3.**

Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, given that the roots are in arithmetical progression.

**Solution:**

Denote the roots by $\alpha - d, \alpha$ and $\alpha + d$. Then the sum of the roots is $3\alpha$, the sum of the products of the roots two at a time is $3\alpha^2 - d^2$, and the product of the roots is $\alpha(\alpha^2 - d^2)$. Hence we have the equations $3\alpha = 6, 3\alpha^2 - d^2 = \frac{23}{4}$ and $\alpha(\alpha^2 - d^2) = -\frac{9}{4}$. From the first equation, we find $\alpha = 2$, and from the second $d = \pm\frac{5}{2}$. Since these values satisfy the third, the three equations are consistent. Thus the roots are $-\frac{1}{2}$, 2 and $\frac{9}{2}$.

**Example 4.**

Solve the equation $24x^3 - 14x^2 - 63x + 45 = 0$, one root being double another.

**Solution:**

Denote the roots by $\alpha, 2\alpha$ and $\beta$. Then we have $3\alpha + \beta = \frac{7}{12}, 2\alpha^2 + 3\alpha\beta = -\frac{21}{8}$ and $2\alpha^2 b = -\frac{15}{8}$. From the first two equations, we obtain $0 = 8\alpha^2 - 2\alpha - 3 = (4\alpha - 3)(2\alpha + 1)$, so that $\alpha = \frac{3}{4}$ or $-\frac{1}{2}$. However, when $\alpha = -\frac{1}{2}$,

$\beta = \frac{25}{12}$, but these values do not satisfy the third equation. Hence the roots are $\alpha = \frac{3}{4}, 2\alpha = \frac{3}{2}$ and $\beta = -\frac{5}{3}$.

Although we may not be able to, without extra information, find the roots of an equation in this way, we can make use of the relations to determine the values of certain functions of the roots. In particular we can evaluate functions for which an interchange of roots has no impact. Such functions, for example $\alpha^2 + \beta^2$ and $\alpha\beta^2 + \beta\alpha^2$, are called symmetric funtions.

**Example 5.**

Find the sum of the squares and of the cubes of the roots of the equation $x^3 - px^2 + qx - r = 0$.

**Solution:**

Denote the roots by $\alpha, \beta$ and $\gamma$. Then $\alpha+\beta+\gamma = p$ and $\beta\gamma+\gamma\alpha+\alpha\beta = q$. Now $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\beta\gamma+\gamma\alpha+\alpha\beta) = p^2-2q$. Substitute $\alpha, \beta$ and $\gamma$ for $x$ in the given equation. Addition yields $\alpha^3+\beta^3+\gamma^3 = -p(\alpha^2+\beta^2+\gamma^2)+q(\alpha+\beta+\gamma)-3r = 0$. Hence $\alpha^3+\beta^3+\gamma^3 = p(p^2-2q)-pq+3r = p^3-3pq+3r$.

**Rational Root Theorem**

The rational roots, if any, of the polynomial $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$ have for their numerator a factor of $p_n$ and for their denominator a factor of $p_0$. These ratios may be positive or negative. It is not the case that every such ratio is a root.

**Proof:**

Suppose the polynomial function $f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$ has a rational root $\frac{a}{b}$ expressed in lowest terms. Then

$$f\left(\frac{a}{b}\right) = p_0\left(\frac{a}{b}\right)^n + p_1\left(\frac{a}{b}\right)^{n-1} + p_2\left(\frac{a}{b}\right)^{n-2} + \cdots + p_{n-1}\left(\frac{a}{b}\right) + p_n = 0$$

Multiplying throughout by $b^n$ we have,

$$\begin{array}{rcl} p_0a^n + bp_1a^{n-1} + b^2p_2a^{n-2} + \cdots + b^{n-1}p_{n-1}a + b^np_n & = & 0 \\ p_0a^n + b\left(p_1a^{n-1} + bp_2a^{n-2} + \cdots + b^{n-2}p_{n-1}a + b^{n-1}p_n\right) & = & 0 \end{array}$$

Since $b$ is a factor of the second term it must be a factor of $p_0a^n$. However, $b$ is not a factor of $a^n$ since the $\frac{a}{b}$ is in lowest terms. Hence $b$ is a factor of $p_0$.

Similarly, we have

$$a\left(p_0a^{n-1} + bp_1a^{n-2} + b^2p_2a^{n-3} + \cdots + b^{n-1}p_{n-1}\right) + b^np_n = 0.$$

Since $a$ is a factor of the first term it must be a factor of $b^np_n$. However, $a$ is not a factor of $b^n$ since the $\frac{a}{b}$ is in lowest terms. Hence $a$ is a factor of $p_n$.

**Example 6.**

Find the roots of $f(x) = 2x^3 - 7x^2 + 2x + 3$ using the rational root theorem.

**Solution:**

The list of potential rational roots is $\pm 1, \pm 3, \pm \frac{3}{2}, \pm \frac{1}{2}$. By trial we find that $f(1) = 0$ so 1 is a root and $x - 1$ is a factor. We could look for more rational roots, but it will be easier to divide by each factor as we find it. We have $f(x) = (x-1)(2x^2 - 5x - 3) = (x-1)(2x+1)(x-3)$ so the roots are 1, $-\frac{1}{2}$, and 3.

Next are two important results on the roots of equations.

**Conjugate Imaginary Roots Theorem.**
In a polynomial equation with real coefficients, imaginary roots occur in conjugate pairs.

**Proof:**

Suppose that $f(x) = 0$ is a polynomial equation with real coefficients, and suppose that it has an imaginary root $a + bi$, $b \neq 0$. We shall show that $a - bi$ is also a root.

The factor of $f(x)$ corresponding to these two roots is $(x - a - bi)(x - a + bi) = (x-a)^2 + b^2$. Let $f(x)$ be divided by $(x-a)^2 + b^2$. Denote the quotient by $Q(x)$, and the remainder, if any, by $Rx + R'$. Then $f(x) = Q(x)((x-a)^2 + b^2) + Rx + R'$.

In this identity put $x = a + bi$. Then $f(x) = 0$ by hypothesis. Also, $(x-a)^2 + b^2 = 0$. Hence $R(a + bi) + R' = 0$. Equating to zero the real and imaginary parts, $Ra + R' = 0$ and $Rb = 0$. Since $b \neq 0$ by hypothesis, we have $R = 0$ and $R' = 0$. Hence $f(x)$ is divisible by $(x-a)^2 + b^2$, that is, by $(x - a - bi)(x - a + bi)$. Hence $x = a - bi$ is also a root.

In the preceding result, we have seen that if the equation $f(x) = 0$ has a pair of conjugate imaginary roots $a \pm bi$, then $(x-a)^2 + b^2$ is a factor of the expression $f(x)$. Suppose that $a \pm bi$, $c \pm di$, $e \pm gi$, ... are the imaginary roots of the equation $f(x) = 0$, and that $\Phi(x)$ is the product of the quadratic factors corresponding to these imaginary roots; then

$$\Phi(x) = ((x-a)^2 + b^2)((x-c)^2 + d^2)((x-e)^2 + g^2) + \cdots .$$

Now each of these factors is positive for every real value of $x$. Hence $\Phi(x)$ is always positive for real values of $x$.

Using the same technique as in the preceding proof, we can establish the following result.

**Conjugate Quadratic Irrational Roots Theorem.**
In a polynomial equation with rational coefficients, quadratic irrational roots occur in conjugate pairs.

In other words, if $a + \sqrt{b}$ is a root, then $a - \sqrt{b}$ is also a root.

**Example 7.**
Solve the equation $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, having given that one root

is $2-\sqrt{3}$.

**Solution:**

Since $2-\sqrt{3}$ is a root, we know that $2+\sqrt{3}$ is also a root. Corresponding to this pair of roots, we have the quadratic factor $x^2-4x+1$. Also, $6x^4-13x^3-35x^2-x+3=(x^2-4x+1)(6x^2+11x+3)$. Hence the other roots are obtained from $0=6x^2+11x+3=(3x+1)(2x+3)$. Thus the roots are $-\frac{1}{3}$, $-\frac{3}{2}$ and $2\pm\sqrt{3}$.

**Example 8.**
Form the equation of the fourth degree with rational coefficients, one of whose roots is $\sqrt{2}+\sqrt{3}i$.

**Solution:**

Here we must have $\sqrt{2}\pm\sqrt{3}i$ as one pair of roots, and $-\sqrt{2}\pm\sqrt{3}i$ as another pair. Corresponding to the first pair we have the quadratic factor $x^2-2\sqrt{2}x+5$, and corresponding to the second pair we have the quadratic factor $x^2+2\sqrt{2}x+5$. Thus the required equation is

$$(x^2+2\sqrt{2}x+5)(x^2-2\sqrt{2}x+5)=0,$$

which may be rewritten as $(x^2+5)^2-8x^2=0$ or $x^4+2x^2+25=0$.

**Example 9.**
Show that the equation $\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}+\cdots+\frac{H}{x-h}=k$ has no imaginary roots.

**Solution:**

If there is an imaginary root $p+qi$, then $p-qi$ is also a root. Substitute these values for $x$ and subtract the first result from the second. The result is

$$q\left(\frac{A^2}{(p-a)^2+q^2}+\frac{B^2}{(p-b)^2+q^2}+\frac{C^2}{(p-c)^2+q^2}+\cdots+\frac{H^2}{(p-h)^2+q^2}\right)=0,$$

which is impossible unless $q=0$.

**Example 10.**
Prove that every cubic polynomial with real coefficients has a real root.

**Solution:**

By the Fundamental Theorem there are three complex roots. Suppose that all three are imaginary. Then two of them are conjugate, but the third cannot have a conjugate. Hence the third is real. Note that there also cannot be only one imaginary root: there are either two or no imaginary roots to a cubic.

### Other Insights into the Roots of Equations

To determine the nature of some of the roots of an equation, it is not always necessary to solve it. For instance, the truth of the following statements will be readily admitted.

- If the coefficients are all positive, the equation has no positive root. Thus the equation $x^4 + x^3 + 2x + 1 = 0$ cannot have a positive root.

- If the coefficients of the even powers of $x$ are all of one sign, and the coefficients of the odd powers are all of the contrary sign, the equation has no negative root. Thus the equation $x^7+x^5-2x^4+x^3-3x^2+7x-5 = 0$ cannot have a negative root.

- If the equation contains only even powers of $x$ and the coefficients are all of the same sign, the equation has no real root. Thus the equation $2x^8 + 3x^4 + x^2 + 7 = 0$ cannot have a real root.

- If the equation contains only odd powers of $x$, and the coefficients re all of the same sign, the equation has no real root except $x = 0$. Thus the equation $x^9 + 2x^5 + 3x^3 + x = 0$ has no real root except $x = 0$.

All the foregoing results are included in the next result.

**Descartes' Rule of Signs.**
A polynomial equation $f(x) = 0$ cannot have more positive roots than there are changes of sign in $f(x)$, and cannot have more negative roots than there are changes of sign in $f(-x)$.

**Proof:**

Suppose that the signs of the terms in a polynomial are

$$+ \quad + \quad - \quad - \quad + \quad - \quad - \quad - \quad + \quad - \quad + \quad -.$$

We shall show that if this polynomial is multiplied by a binomial whose signs are + -, there will be at least one more change of sign in the product than in the original polynomial. Writing down only the signs of the terms in the multiplication, we have:

$$\begin{array}{ccccccccccccc}
 & + & + & - & - & + & - & - & - & + & - & + & - \\
 & & & & & & & & & & & + & - \\
\hline
 & - & - & + & + & - & + & + & + & - & + & - & + \\
+ & + & - & - & + & - & - & - & + & - & + & - & \\
\hline
+ & \pm & - & \mp & + & - & \mp & \mp & + & - & + & - & +.
\end{array}$$

Hence we see that in the product:

(i) an ambiguity replaces each continuation of sign in the original polynomial;

(ii) the signs before and after an ambiguity or set of ambiguities are unlike;

(iii) a change of sign is introduced at the end.

Let us take the most unfavorable case and suppose that all the ambiguities are replaced by continuations; from (ii) we see that the number of changes of sign will be the same whether we take the upper or the lower signs; let us take the upper; thus the number of changes of sign cannot be less than in

$$+ \quad \pm \quad - \quad \mp \quad + \quad - \quad \mp \quad \mp \quad + \quad - \quad + \quad - \quad +.$$

and this series of signs is the same as in the original polynomial with an additional chance of sign at the end. If we then suppose the factors corresponding to the negative and imaginary roots to be already multiplied together, each factor $x - a$ corresponding to a positive root introduces at least one change of sign. Therefore no equation can have more positive roots than it has changes of sign.

Again, the roots of the equation $f(-x) = 0$ are equal to those of $f(x) = 0$ but opposite to them in sign. Therefore the negative roots of $f(x) = 0$ are the positive roots of $f(-x) = 0$. The number of these positive roots cannot exceed the number of changes of sign in $f(-x)$, that is, the number of negative roots of $f(x) = 0$ cannot exceed the number of changes of sign in $f(-x)$.

**Example 11.**
Find the nature of the roots of the equation $f(x) = x^9 + 5x^8 - x^3 + 7x + 2 = 0$.

**Solution:**

There are two changes of sign in $f(x)$. Therefore there are at most two positive roots. Now $f(-x) = -x^9 + 5x^8 + x^3 - 7x + 2$ has three changes of sign. Therefore the given equation has at most three negative roots. Hence it must have at least four imaginary roots.

### Transformations

The discussion of an equation is sometimes simplified by transforming it into another equation whose roots bear some assigned relation to those of the one proposed. Such transformations are especially useful in the solution of cubic equations.

Suppose we wish to transform an equation into another whose roots are those of the proposed equation with contrary signs. Let $f(x) = 0$ be the proposed equation. Put $-y$ for $x$. Then the equation $f(-y) = 0$ is satisfied by every root of $f(x) = 0$ with its sign changed. Thus the required equation is $f(-y) = 0$. If the proposed equation is $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$, then the required equation will be $p_0y^n - p_1y^{n-1} + p_2y^{n-2} - \cdots + (-1)^{n-1}p_{n-1}y + (-1)^n p_n = 0$, which is obtained from the original equation by changing the sign of every alternate term beginning with the second.

Suppose we wish to transform an equation into another whose roots are equal to those of the proposed equation multiplied by a given quantity. Let $f(x) = 0$ be the proposed equation, and let $q$ denote the given quantity. Put $y = qx$, so that $x = \frac{y}{q}$. Then the required equation is $f(\frac{y}{q}) = 0$. The chief use of this transformation is to clear an equation of fractional coefficients.

**Example 1.**
Remove fractional coefficients from the equation $2x^3 - \frac{3}{2}x^2 - \frac{1}{8}x + \frac{3}{16} = 0$.

**Solution:**

Put $x = \frac{y}{q}$ and multiply each term by $q^3$. Thus $2y^3 - \frac{3}{2}qy^2 - \frac{1}{8}q^2y + \frac{3}{16}q^3 = 0$. By putting $q = 4$, all the terms become integral, and on dividing by 2, we obtain $y^3 - 3y^2 - y + 6 = 0$.

Suppose we wish to find the equation whose roots are the squares of those of a proposed equation. Let $f(x) = 0$ be the given equation. Putting $y = x^2$, we have $x = \sqrt{y}$. Hence the required equation is $f(\sqrt{y}) = 0$.

**Example 2.**
Find the equation whose roots are the squares of those of the equation $x^3 + px^2 + qx + r = 0$.

**Solution:**

Putting $x = \sqrt{y}$ and transposing, we have $(y+q)\sqrt{y} = -(py+r)$. Squaring both sides, we have $(y^2 + 2qy + q^2)y = p^2y^2 + 2pry + r^2$ or $y^3 + (2q - p^2)y^2 + (q^2 - 2pr)y - r^2 = 0$.

**Remark:**
Compare the solution given in Chapter VII.

**Reciprocal Equations**
Suppose we wish to transform an equation into another whose roots are the reciprocals of the roots of the proposed equation. Let $f(x) = 0$ be the proposed equation. Put $y = \frac{1}{x}$ so that $x = \frac{1}{y}$. Then the required equation is $f(\frac{1}{y}) = 0$. One of the chief uses of this transformation is to obtain the values of expressions which involve symmetrical functions of negative powers of the roots.

**Example 3.**
If $\alpha$, $\beta$ and $\gamma$ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$.

**Solution:**

Write $\frac{1}{y}$ for $x$, multiply by $y^3$ and change all the signs. Then the resulting equation will be $ry^3 - qy^2 + py - 1 = 0$, with roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$. Hence $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{q}{r}$, $\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} = \frac{p}{r}$ so that $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{q^2-2pr}{r^2}$.

**Example 4.**
If $\alpha$, $\beta$ and $\gamma$ are the roots of $x^3 + 2x^2 - 3x - 1 = 0$, find the value of $\alpha^{-3} + \beta^{-3} + \gamma^{-3}$.

**Solution:**

Writing $\frac{1}{y}$ for $x$, the transformed equation is $y^3+3y^2-2y-1=0$. The given expression is equal to the value of $S_3$ in this equation. Here $S_1=-3$, $S_2=(-3)^2-2(-2)=13$ and $S_3+3S_2-2S_1-3=0$. Hence $S_3=-42$.

If an equation is unaltered by changing $x$ into $\frac{1}{x}$, it is called a *reciprocal* equation. If the given equation is $x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-2}x^2+p_{n-1}x+p_n=0$, the equation obtained by writing $\frac{1}{x}$ for $x$ and clearing of fractions is $p_nx^n+p_{n-1}x^{n-1}+p_{n-2}x^{n-2}+\cdots+p_2x^2+p_1x+1=0$. If these two equations are the same, we must have $p_1=\frac{p_{n-1}}{p_n}$, $p_2=\frac{p_{n-2}}{p_n}$, $\ldots$, $p_{n-2}=\frac{p_2}{p_n}$, $p_{n-1}=\frac{p_1}{p_n}$ and $p_n=\frac{1}{p_n}$.

From the last result we have $p_n=\pm1$, and thus we have two classes of reciprocal equations.

(i) If $p_n=1$, then $p_1=p_{n-1}$, $p_2=p_{n-2}$, $p_3=p_{n-3}$, $\ldots$; in other words, the coefficients of terms equidistant from the beginning and end are equal.

(ii) If $p_n=-1$, then $p_1=-p_{n-1}$, $p_2=-p_{n-2}$, $p_3=-p_{n-3}$, $\ldots$. Hence if the equation is of degree $2m$, then $p_m=-p_m$ or $p_m=0$. In this case the coefficients of terms equidistant from the beginning and end are equal in magnitude and opposite in sign, and if the equation is of an even degree, the middle term is missing.

Suppose that $f(x)=0$ is a reciprocal equation. If it is of the first class and of an odd degree, it has a root $-1$, so that $f(x)$ is divisible by $x+1$. If $Q(x)$ is the quotient, then $Q(x)=0$ is a reciprocal equation of the first class and of an even degree.

If $f(x)=0$ is of the second class and of an odd degree, it has a root 1, so that $f(x)$ is divisible by $x-1$. As before, $Q(x)=0$ is a reciprocal equation of the first class and of an even degree. If $f(x)=0$ is of the second class and of an even degree, it has a root 1 and a root $-1$, so that $f(x)$ is divisible by $x^2-1$. As before, $Q(x)=0$ is a reciprocal equation of the first class and of an even degree.

Hence any reciprocal equation is of an even degree with its last term positive, or can be reduced to this form, which may therefore be considered as the standard form of reciprocal equations.

A reciprocal equation of the standard form can be reduced to an equation of half its degree. Let the equation be $p_0x^{2m}+p_1x^{2m-1}+p_2x+2m-2+\cdots+p_mx^m+\cdots+p_2x^2+p_1x+p_0=0$. Dividing by $x^m$ and rearranging the terms, we have $p_0(x^m+\frac{1}{x_m})+p_1(x^{m-1}+\frac{1}{x^{m-1}})+p_3(x^{m-2}+\frac{1}{x^{m-2}})+\cdots+p_m=0$. Now $x^{k+1}+\frac{x^{k+1}}{=}(x^k+\frac{1}{x^k})(x+\frac{1}{x})-(x^{k-1}+\frac{1}{x^{k-1}})$.

Letting $z=x+\frac{1}{x}$ and giving in succession to $k$ the values 1, 2, 3, $\ldots$, we obtain $x^2+\frac{1}{x^2}=z^2-2$, $x^3+\frac{1}{x^3}=z(z^2-2)-z=z^3-3z$, $x^4+\frac{1}{x^4}=$

$z(z^3-3z)-(z^2-2)=z^4-4z^2+2$, and so on. Generally $x^m+\frac{1}{x^m}$ is of degree $m$ in $z$, and therefore the equation in $z$ is of degree $m$.

**Linear Shifts**

Suppose we wish to transform an equation into another whose roots exceed those of the proposed equation by a given quantity. Let $f(x)=0$ be the proposed equation, and let $h$ be the given quantity. Put $y=x+h$ so that $x=y-h$. Then the required equation is $f(y-h)=0$. Similarly $f(y+h)=0$ is an equation whose roots are less by $h$ than those of $f(x)=0$.

**Example 5.**

Find the equation whose roots exceed by 2 the roots of the equation $4x^4+32x^3+83x^2+76x+21=0$.

**Solution:**

The required equation will be obtained by substituting $y-2$ for $x$ in the proposed equation. We have

$$\begin{aligned} & 4(y-2)^4+32(y-2)^3+83(y-2)^2+76(y-2)+21 \\ = \; & 4(y^4-8y^3+24y^2-32y+16)+32(y^3-6y^2+12y-8) \\ & +83(y^2-4y+4)+76(y-2)+21 \\ = \; & 4y^4-13y^2+9. \end{aligned}$$

**Remark:**

From $0=4y^4-13y^2+9=(4y^2-9)(y^2-1)=(2y+3)(2y-3)(y+1)(y-1)$, the roots of this equation are $\pm\frac{3}{2}$ and $\pm 1$. Hence the root of the proposed equation are $-\frac{1}{2}$, $-\frac{7}{2}$, $-1$ and $-3$.

The chief use of this linear shift is to remove some assigned term from an equation. Let the given equation be $p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0$. If $y=x-h$, we obtain the new equation

$$p_0(y+h)^n+p_1(y+h)^{n-1}+p_2(y+h)x^{n-2}+\cdots+p_n=0.$$

When arranged in descending powers of $y$, it becomes $p_0y^n+(np_0h+p_1)y^{n-1}+(\frac{n(n-1)}{2}p_0h^2+(n-1)p_1h+p_2)y^{n-2}+\cdots$.

If the term to be removed is the second, we put $np_0h+p_1=0$, so that $h=\frac{-p_1}{np_0}$. If the term to be removed is the third we put

$$\frac{n(n-1)}{2}p_0h^2+(n-1)p_1h+p_2=0,$$

and so obtain a quadratic to find $h$. Similarly, we may remove any other assigned term. Sometimes it will be more convenient to proceed in a different way.

**Example 6.**

Remove the second term from the equation $px^3+qx^2+rx+s=0$.

**Solution:**

Let $\alpha$, $\beta$ and $\gamma$ be the roots, so that $\alpha+\beta+\gamma=-\frac{q}{p}$. If we increase each of the roots by $\frac{q}{3p}$, then in the transformed equation the sum of the roots will be equal to $-\frac{q}{p}+\frac{q}{p}=0$. In other words, the coefficient of the second term will be zero. Hence the required transformation will be effected by substituting $y-\frac{q}{3p}$ for $x$ in the given equation. It is $py^3+(r-\frac{q^2}{3p})y+(\frac{2q^3}{27p^2}-\frac{qr}{3p}+s)=0$.

We have seen many examples of how, from the equation $f(x)=0$, we may form an equation whose roots are connected with those of the given equation by some assigned relation. Let $y$ be a root of the required equation. Then the transformed equation can usually be obtained by expressing $x$ as a function of $y$ by means of the assigned relation and substituting this value of $x$ in $f(x)=0$. Sometimes, it is necessary to eliminate $x$ between $f(x)=0$ and the assigned relation.

**Example 7.**

If $\alpha$, $\beta$ and $\gamma$ are the roots of the equation $x^3+px^2+qx+r=0$, form the equation whose roots are $\alpha-\frac{1}{\beta\gamma}$, $\beta-\frac{1}{\gamma\alpha}$ and $\gamma-\frac{1}{\alpha\beta}$.

**Solution:**

When $x=\alpha$ in the given equation, $y=\alpha-\frac{1}{\beta\gamma}$ in the transformed equation. Note that we have $\alpha-\frac{1}{\beta\gamma}=\alpha-\frac{\alpha}{\alpha\beta\gamma}=\alpha+\frac{\alpha}{r}$. Therefore the transformed equation will be obtained by the substitution $y=x+\frac{x}{r}$ or $x=\frac{ry}{1+r}$. Thus the required equation is $r^2y^3+pr(1+r)y^2+q(1+r)^2y+(1+r)^3=0$.

**Example 8.**

Form the equation whose roots are the squares of the differences of the roots of $x^3+qx+r=0$.

**Solution:**

Let $\alpha$, $\beta$ and $\gamma$ be the roots of the cubic. Then the roots of the required equation are $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$ and $(\alpha-\beta)^2$. Now

$$\begin{aligned}(\beta-\gamma)^2 &= \beta^2+\gamma^2-2\beta\gamma\\ &= \alpha^2+\beta^2+\gamma^2-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha}\\ &= (\alpha+\beta+\gamma)^2-2(\beta\gamma+\gamma\alpha+\alpha\beta)-\alpha^2-\frac{2\alpha\beta\gamma}{\alpha}\\ &= -2q-\alpha^2+\frac{2r}{\alpha}.\end{aligned}$$

When $x=\alpha$ in the given equation, $y=(\beta-\gamma)^2$ in the transformed equation. It follows that $y=-2q-x^2+\frac{2r}{x}$. Thus we have to eliminate $x$ between the equations $x^3+qx+r=0$ and $x^3+(2q+y)x-2r=0$. By subtraction, $(q+y)x=3r$ or $x=\frac{3r}{q+y}$. Substituting and reducing, we obtain $y^3+6qy^2+9q^2y+(27r^2+4q^3)=0$.

**Remark:**

If $\alpha$, $\beta$ and $\gamma$ are real, $(\beta - \gamma)^2$, $(\gamma - \alpha)^2$ and $(\alpha - \beta)^2$ are all non-negative. Therefore, $27r^2 + 4q^3$ is non-positive. Hence in order that the equation $x^3 + qx + r = 0$ may have all its roots real, $27r^2 + 4q^3$ must be non-positive. In other words, $\frac{r^2}{4} + \frac{q^3}{27}$ must be non-positive.

If $27r^2 + 4q^3 = 0$, the transformed equation has one root zero. Therefore the original equation has two equal roots. If $27r^2 + 4q^3$ is positive, the transformed equation has a negative root. Therefore the original equation must have two imaginary roots, since it is only such a pair of roots which can produce a negative root in the transformed equation.

## EXERCISES IV

Form the equation whose roots are:

1. $\frac{2}{3}, \frac{3}{2}, \pm\sqrt{3}$.

2. $0, 0, 2, 2, -3, -3$.

3. $2, 2, -2, -2, 0, 5$.

4. $\alpha + \beta, \alpha - \beta, -\alpha + \beta, -\alpha - \beta$.

Solve the equations:

5. $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$, two roots being 1 and 7.
6. $4x^3 + 16x^2 - 9x - 36 = 0$, the sum of two of the roots being zero.
7. $4x^3 + 20x^2 - 23x + 6 = 0$, two of the roots being equal.
8. $3x^3 - 26x^2 + 52x - 24 = 0$, the roots being in geometrical progression.
9. $2x^3 - x^2 - 22x - 24 = 0$, two of the roots being in the ratio of 3:4.
10. $24x^3 + 46x^2 + 9x - 9 = 0$, one root being double another of the roots.
11. $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$, two of the roots being equal but opposite in sign.
12. $54x^3 - 39x^2 - 26x + 16 = 0$, the roots being in geometrical progression.
13. $32x^3 - 48x^2 + 22x - 3 = 0$, the roots being in arithmetical progression.
14. $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$, the product of two of the roots being 2.
15. $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, the roots being in arithmetical progression.
16. $27x^4 - 195x^3 + 494x^2 - 520x + 192 = 0$, the roots being in geometrical progression.

17. $18x^3 + 8lx^2 + 121x + 60 = 0$, one root being half the sum of the other two.

18. If $\alpha, \beta$ and $\gamma$ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the value of

    (a) $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$;

    (b) $\frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2} + \frac{1}{\alpha^2\beta^2}$.

19. If $\alpha, \beta$ and $\gamma$ are the roots of $x^3 + qx + r = 0$, find the value of

    (a) $(\beta - \gamma)^2 + (\gamma - \alpha)^2 + (\alpha - \beta)^2$;

    (b) $(\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1} + (\alpha + \beta)^{-1}$.

20. Find the sum of the squares and of the cubes of the roots of $x^4 + qx^2 + rx + 8 = 0$.

21. Find the sum of the fourth powers of the roots of $x^3 + qx + r = 0$.

22. Prove the Conjugate Quadratic Irrational Roots Theorem.

Solve the equations:

23. $3x^4 - 10x^3 + 4x^2 - x - 6 = 0$, one root being $\frac{1+\sqrt{3}}{2}$.

24. $6x^4 - 13x^3 - 35x^2 - x + 3 = 0$, one root being $2 - \sqrt{3}$.

25. $x^4 + 4x^3 + 5x^2 + 2x - 2 = 0$, one root being $-1 + i$.

26. $x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$, one root being $i$.

27. $x^5 - x^4 + 8x^2 - 9x - 15 = 0$, one root being $\sqrt{3}$ and another $1 - 2i$.

Form the equation of lowest degrees with rational coefficients, one of whose roots is:

28. $\sqrt{3} + \sqrt{2}i$. 29. $\sqrt{5} - i$. 30. $-\sqrt{2} - \sqrt{2}i$. 31. $\sqrt{5 + 2\sqrt{6}}$.

Form the equation whose roots are:

32. $\pm 4\sqrt{3}, 5 \pm 2i$. 33. $1 \pm \sqrt{2}i, 2 \pm \sqrt{3}i$.

34. Form the equation of the eighth degree with rational coefficients, one of whose roots is $\sqrt{2} + \sqrt{3} + i$.

35. Find the nature of the roots of the equation $3x^4 + 12x^2 + 5x - 4 = 0$.

36. Show that the equation $2x^7 - x^4 + 4x^3 - 5 = 0$ has at least four imaginary roots.

37. What may be inferred respecting the roots of the equation $x^{10} - 4x^6 + x^4 - 2x - 3 = 0$?

38. Find the least possible number of imaginary roots of the equation $x^9 - x^5 + x^4 + x^2 + 1 = 0$.

39. Transform the equation $x^3 - 4x^2 + \frac{1}{4}x - \frac{1}{5} = 0$ into another with integral coefficients, and unity for the coefficient of the first term.

40. Transform the equation $3x^4 - 5x^3 + x^2 - x + 1 = 0$ into another the coefficient of whose first term is unity.

Solve the equations:

37. $2x^4 + x^3 - 6x^2 + x + 2 = 0$.

38. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

39. $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.

40. $4x^6 - 24x^5 + 57x^4 - 73x^3 + 57x^2 - 24x + 4 = 0$.

41. Solve the equation $3x^3 - 22x^2 + 48x - 32 = 0$ the roots of which are in harmonical progression.

42. The roots of $x^3 - 11x^2 + 36x - 36 = 0$ are in harmonical progression. Find them.

43. If the roots of the equation $x^3 - ax^2 + x - b = 0$ are in harmonical progression, show that the mean root is $3b$.

44. Solve the equation $40x^4 - 22x^3 - 21x^2 + 2x + 1 = 0$, the roots of which are in harmonical progression.

Remove the second term from the equations:

45. $x^3 - 6x^2 + 10x - 3 = 0$.

46. $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$.

47. $x^5 + 5x^4 + 3x^3 + x^2 + x - 1 = 0$.

48. $x^6 - 12x^5 + 3x^2 - 17x + 300 = 0$.

49. Transform the equation $x^3 - \frac{1}{4}x - \frac{3}{4} = 0$ into one whose roots exceed by 3 the corresponding roots of the given equation.

50. Diminish by 3 the roots of the equation $x^5 - 4x^4 + 3x^2 - 4x + 6 = 0$.

51. Find the equation each of whose roots is greater by 1 than the corresponding root of the equation $x^3 - 5x^2 + 6x - 3 = 0$.

52. Find the equation whose roots are the squares of the roots of $x^4 + x^3 + 2x^2 + x + 1 = 0$.

53. Form the equation whose roots are the cubes of the roots of $x^3 + 3x^2 + 2 = 0$.

If $\alpha$, $\beta$ and $\gamma$ are the roots of $x^3 + qx + r = 0$, form the equation whose roots are:

54. $k\alpha - 1, k\beta - 1, k\gamma - 1$.

55. $\beta^2\gamma^2, \gamma^2\alpha^2, \alpha^2\beta^2$.

56. $\frac{\beta+\gamma}{\alpha^2}, \frac{\gamma+\alpha}{\beta^2}, \frac{\alpha+\beta}{\gamma^2}$.

57. $\beta\gamma + \frac{1}{\alpha}, \gamma\alpha + \frac{1}{\beta}, \alpha\beta + \frac{1}{\gamma}$.

58. $\alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta)$.

59. $\alpha^3, \beta^3, \gamma^3$.

60. $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

61. Show that the cubes of the roots of $x^3 + ax^2 + bx + ab = 0$ are given by the equation $x^3 + a^3x^2 + b^3x + a^3b^3 = 0$.

62. Solve the equation $x^5 - 5x^4 - 5x^3 + 25x^2 + 4x - 20 = 0$, whose roots are of the form $\alpha, -\alpha, \beta, -\beta, \gamma$.

63. If the roots of $x^3 + 3px^2 + 3qx + r = 0$ are in harmonical progression, show that $2q^3 = r(3pq - r)$.

## Solutions to Even-numbered Exercises IV

2. We have $0 = x^2(x-2)^2(x+3)^2 = x^2(x^2 - 4x + 4)(x^2 + 6x + 9) = x^6 + 2x^5 - 11x^4 - 12x^3 + 36x^2$.

4. We have

$$\begin{aligned}
0 &= (x - (\alpha + \beta))(x - (\alpha - \beta))(x - (-\alpha + \beta))(x - (-\alpha - \beta)) \\
&= ((x - \alpha)^2 - \beta^2)((x + \alpha)^2 - \beta^2)) \\
&= (x - \alpha)^2(x + \alpha)^2 - \beta^2((x - \alpha)^2 + (x + \alpha)^2) + \beta^4 \\
&= x^4 - 2\alpha^2x^2 + \alpha^4 - 2\beta^2x^2 - 2\alpha^2\beta^2 + \beta^4 \\
&= x^4 - 2(\alpha^2 + \beta^2) + (\alpha^2 - \beta^2)^2.
\end{aligned}$$

6. Let the roots be $\pm\alpha$ and $\beta$. Since $\alpha + (-\alpha) + \beta = \frac{16}{4}$, we have $\beta = 4$. Since $\alpha(-\alpha)\beta = -\frac{36}{4}$, we have $\alpha = \pm\frac{3}{2}$.

8. Let the roots be $\frac{\alpha}{r}$, $\alpha$ and $\alpha r$. Then $(\frac{\alpha}{r})\alpha(\alpha r) = \frac{24}{3}$, so that $\alpha = 2$. Factoring out $x-2$, we have $0 = (x-2)(3x^2-20x+12) = (x-2)(3x-2)(x-6)$. Hence the roots are 2, $\frac{2}{3}$ and 6.

10. Let the roots be $\alpha$, $2\alpha$ and $\beta$. Then $\alpha + 2\alpha + \beta = -\frac{23}{12}$, $\alpha(2\alpha) + \alpha\beta + (2\alpha)\beta = \frac{3}{8}$ and $\alpha(2\alpha)\beta = \frac{3}{8}$. From the first equation, $\beta = -3\alpha - \frac{23}{12}$. Substituting into the second equation, we have $2\alpha^2 + 3\alpha(-3\alpha - \frac{23}{12}) = \frac{3}{8}$. This is equivalent to $0 = 56\alpha^2+46\alpha+3 = (14\alpha+1)(4\alpha+3)$. If $\alpha = -\frac{1}{14}$, then $\beta = \frac{3}{14} - \frac{23}{12} = -\frac{143}{84}$, but $2\alpha^2\beta \neq \frac{3}{8}$. If $\alpha = -\frac{3}{4}$, then $\beta = \frac{9}{4} - \frac{23}{12} = \frac{1}{3}$, and the third equation is satisfied this time. Hence the roots are $\frac{3}{4}$, $-\frac{3}{2}$ and $\frac{1}{3}$.

12. Let the roots be $\frac{\alpha}{r}$, $\alpha$ and $\alpha r$. Then $(\frac{\alpha}{r})\alpha(\alpha r) = -\frac{8}{27}$, so that $\alpha = -\frac{2}{3}$. Factoring out $3x+2$, we have $0 = (3x+2)(18x^2-25x+8) = (3x+2)(9x-8)(2x-1)$. Hence the roots are $\frac{8}{9}$, $-\frac{2}{3}$ and $\frac{1}{2}$.

14. Let the roots be $\alpha$, $\beta$, $\gamma$ and $\delta$ with $\alpha\beta = 2$. Then $\gamma\delta = -1$. We have $(\alpha+\beta)+(\gamma+\delta) = \frac{29}{6}$ while $\frac{7}{6} = \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = 2(\gamma+\delta)-(\alpha+\beta)$. Addition yields $3(\gamma+\delta) = 6$ so that $\gamma+\delta = 2$, and subtraction yields $\alpha+\beta = \frac{17}{6}$. Now $\alpha$ and $\beta$ are the roots of $x^2 - \frac{17}{6}x + 2 = 0$. This is equivalent to $0 = 6x^2 - 17x + 12 = (2x-3)(3x-4)$. Hence they are $\frac{3}{2}$ and $\frac{4}{3}$. On the other hand, $\gamma$ and $\delta$ are the roots of $x^2 - 2x - 1 = 0$. Hence they are equal to $\frac{2\pm\sqrt{4+4}}{2} = 1 \pm \sqrt{2}$.

16. Let the roots be $\alpha < \beta < \gamma < \delta$ with $\alpha\delta = \beta\gamma$. Since $\alpha\beta\gamma\delta = \frac{64}{9}$, we have $\alpha\delta = \beta\gamma = \frac{8}{3}$. Now $(\alpha+\delta)+(\beta+\gamma) = \frac{65}{9}$ while $\frac{494}{27} = \alpha\delta+\beta\gamma+\alpha\beta+\alpha\gamma+\beta\delta+\gamma\delta = \frac{16}{3}+(\alpha+\delta)(\beta+\gamma)$ so that $(\alpha+\delta)(\beta+\gamma) = \frac{350}{27}$. It follows that $\alpha+\delta$ and $\beta+\gamma$ are the roots of $x^2 - \frac{65}{9}x + \frac{350}{27} = 0$. This is equivalent to $0 = 27x^2 - 65x + 350 = (9x-35)(3x-10)$. Since $\frac{10}{3} < \frac{35}{9}$, $\beta+\gamma = \frac{10}{3}$ while $\alpha+\delta = \frac{35}{9}$. Note that $\beta$ and $\gamma$ are the roots of $x^2-\frac{10}{3}x+\frac{8}{3} = 0$. This is equivalent to $0 = 3x^2-10x+8 = (3x-4)(x-2)$. Hence $\beta = \frac{4}{3}$ and $\gamma = 2$. On the other hand, $\alpha$ and $d$ are the roots of $x^2-\frac{35}{9}x+\frac{8}{3} = 0$. This is equivalent to $0 = 9x^2-35x+24 = (9x-8)(x-3)$. Hence $\alpha = \frac{9}{8}$ and $\delta = 3$.

18. (a) Note that $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma) = q^2-2pr$. Hence $\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2} = \frac{q^2-2pr}{r^2}$.

    (b) Note that $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha) = p^2-2q$. It follows that $\frac{1}{\beta^2\gamma^2}+\frac{1}{\gamma^2\alpha^2}+\frac{1}{\alpha^2\beta^2} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2\beta^2\gamma^2} = \frac{p^2-2q}{r^2}$.

20. Let the roots be $\alpha$, $\beta$, $\gamma$ and $\delta$. Note that $\alpha+\beta+\gamma+\delta = 0$. Now

$$\alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2-2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta) = -2q.$$

Hence

$$0 = (\alpha+\beta+\gamma+\delta)^3$$

$$\begin{aligned} &= (\alpha^3+\beta^3+\gamma^3+\delta^3)+6(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta) \\ &\quad +3(\alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\beta^2\gamma+\beta^2\delta+\beta^2\alpha+\gamma^2\delta+\gamma^2\alpha \\ &\quad +\gamma^2\beta+\delta^2\alpha+\delta^2\beta+\delta^2\gamma) \\ &= -2(\alpha^3+\beta^3+\gamma^3+\delta^3)+6(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta) \\ &\quad +3(\alpha+\beta+\gamma+\delta)(\alpha^2+\beta^2+\gamma^2+\delta^2) \\ &= -2(\alpha^3+\beta^3+\gamma^3+\delta^3)+6(\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta). \end{aligned}$$

Hence $\alpha^3+\beta^3+\gamma^3+\delta^3=-3r$.

24. We must have $2+\sqrt{3}$ as another root. Now $(x-(2-\sqrt{3}))(x-(2+\sqrt{3}))=x^2-4x+1$. Factoring out $x^2-4x+1$, we have $0=(x^2-4x+1)(6x^2+11x+3)=(x^2-4x+1)(3x+1)(2x+3)$. Hence the other two roots are $-\frac{1}{3}$ and $-\frac{3}{2}$.

26. We must have $-i$ as another root. Now $(x+i)(x-i)=x^2+1$. Factoring out $x^2+1$, we have $0=(x^2+1)(x^2+4x+5)$. Hence the other two roots are $\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$.

28. We must have

$$\begin{aligned} 0 &= (x-(\sqrt{3}+\sqrt{2}i))(x-(\sqrt{3}-\sqrt{2}i))(x-(-\sqrt{3}+\sqrt{2}i))(x-(-\sqrt{3}-\sqrt{2}i)) \\ &= ((x-\sqrt{3})^2+2)((x+\sqrt{3})^2+2) \\ &= (x^2+5-2\sqrt{3}x)(x^2+5+2\sqrt{3}x) \\ &= (x+5)^2-(2\sqrt{3}x)^2 \\ &= x^4-2x^2+25. \end{aligned}$$

30. We must have

$$\begin{aligned} 0 &= (x-(\sqrt{2}+\sqrt{2}i))(x-(\sqrt{2}-\sqrt{2}i))(x-(-\sqrt{2}+\sqrt{2}i))(x-(-\sqrt{2}-\sqrt{2}i)) \\ &= ((x-\sqrt{2})^2+2)((x+\sqrt{2})^2+2) \\ &= (x^2+4-2\sqrt{2}x)(x^2+4+2\sqrt{2}x) \\ &= (x^2+4)^2-(2\sqrt{2}x)^2 \\ &= x^4+16. \end{aligned}$$

32. We have

$$\begin{aligned} 0 &= (x-4\sqrt{3})(x+4\sqrt{3})(x-(5+2i))(x-(5-2i)) \\ &= (x^2-48)((x-5)^2)+4) \\ &= x^4-10x^3-19x^2+480x-1392. \end{aligned}$$

34. We must have

$$0 = (x-(\sqrt{2}+\sqrt{3}+i))(x-(\sqrt{2}+\sqrt{3}-i))(x-(\sqrt{2}-\sqrt{3}+i))$$

$$\begin{aligned}
& (x-(\sqrt{2}-\sqrt{3}-i))(x-(-\sqrt{2}+\sqrt{3}+i))(x-(-\sqrt{2}+\sqrt{3}-i)) \\
& (x-(-\sqrt{2}-\sqrt{3}+i))(x-(-\sqrt{2}-\sqrt{3}-i)) \\
= \; & (x-\sqrt{2}-\sqrt{3})^2+1)((x-\sqrt{2}+\sqrt{3})^2+1)((x+\sqrt{2}-\sqrt{3})^2+1) \\
& ((x+\sqrt{2}+\sqrt{3})^2+1) \\
= \; & (x^2-2\sqrt{2}x-2\sqrt{3}x+6+2\sqrt{6}(x^2-2\sqrt{2}x+2\sqrt{3}x+6-2\sqrt{6}) \\
& (x^2+2\sqrt{2}x-2\sqrt{3}x+6-2\sqrt{6})(x^2+2\sqrt{2}x+2\sqrt{3}x+6+2\sqrt{6}) \\
= \; & ((x^2-2\sqrt{2}x+6)^2-(2\sqrt{3}x-2\sqrt{6})^2) \\
& ((x^2+2\sqrt{2}x+6)^2-(2\sqrt{3}x+2\sqrt{6})^2) \\
= \; & (x^4-4\sqrt{2}x^3+20x^2-24\sqrt{2}x+36-12x^2+24\sqrt{2}x-24) \\
& (x^4+4\sqrt{2}x^3+20x^2+24\sqrt{2}x+36-12x^2-24\sqrt{2}x-24) \\
= \; & (x^4+8x^2+12-4\sqrt{2}x^3)(x^4+8x^2+12+4\sqrt{2}x^3) \\
= \; & (x^4+8x^2+12)^2-(4\sqrt{2}x^3)^2 \\
= \; & x^8-16x^6+88x^4+192x^2+144.
\end{aligned}$$

36. Since there are 3 changes of signs, the number of positive roots is at most 3. Letting $x=-y$, we have $-2y^7-y^4-4y^3-5=0$. Since there are no changes of signs, there are no negative roots. It follows that there must be at least 4 imaginary roots.

38. Since there are 2 changes of signs, the number of positive roots is at most 2. Letting $x=-y$, we have $-y^9+y^5+y^4+y^2+1=0$. Since there is only 1 change of signs, the number of negative roots is at most 1. It follows that there must be at least 6 imaginary roots.

40. We have $x^4-\frac{5}{3}x^3+\frac{1}{3}x^2-\frac{1}{3}x+\frac{1}{3}=0$. Let $x=\frac{1}{3}y$. Then $\frac{1}{81}y^4-\frac{5}{81}y^3+\frac{1}{27}y^2-\frac{1}{9}y+\frac{1}{3}=0$ or $y^4-5y^3+3y^2-9y+27=0$.

42. The reciprocal equation may be rewritten as $(x^2+\frac{1}{x^2})-10(x+\frac{1}{x})+26=0$. Let $y=x+\frac{1}{x}$. Then $y^2=x^2+\frac{1}{x^2}+2$. Hence $0=y^2-10y+24=(y-4)(y-6)$ so that $y=4$ or 6. From $x+\frac{1}{x}=4$, we have $x^2-4x+1=0$ so that $x=\frac{4\pm\sqrt{16-4}}{2}=2\pm\sqrt{3}$. From $x+\frac{1}{x}=6$, we have $x^2-6x+1=0$ so that $x=\frac{6\pm\sqrt{36-4}}{2}=3\pm2\sqrt{2}$.

44. The reciprocal equation may be rewritten as $4(x^3+\frac{1}{x^3})-24(x^2+\frac{1}{x^2})+57(x+\frac{1}{x})-73=0$. Let $y=x+\frac{1}{x}$. Then $y^2=x^2+\frac{1}{x^2}+2$ and $y^3=x^3+\frac{1}{x^3}+3(x+\frac{1}{x})$. Hence $4y^3-24y^2+45y-25=0$. By inspection, $y=1$ is a root. From $x+\frac{1}{x}=1$, we have $x^2-x+1=0$ so that $x=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm\sqrt{3}i}{2}$. Factoring out $y-1$, we have $0=(y-1)(4y^2-20y+25)=(y-1)(2y-5)^2$. Hence we have a double root $y=\frac{5}{2}$. From $x+\frac{1}{x}=\frac{5}{2}$, we have $0=2x^2-5x+2=(2x-1)(x-2)$ so that $x=\frac{1}{2}$ or 2. Both of them are double roots.

46. Let the roots be $\alpha$, $\beta$ and their harmonic mean $\frac{2\alpha\beta}{\alpha+\beta}$. From $36=\alpha\beta+\frac{2\alpha^2\beta}{\alpha+\beta}+\frac{2\alpha\beta^2}{\alpha+\beta}=3\alpha\beta$, we have $\alpha\beta=12$. From $\alpha\beta(\frac{2\alpha\beta}{\alpha+\beta})=36$, we have

$\alpha+\beta = 8$. Hence $\alpha$ and $\beta$ are the roots of the equation $0 = x^2-8x+12 = (x-2)(x-6)$. It follows that $\alpha = 2$, $\beta = 6$ and $\frac{2\alpha\beta}{\alpha+\beta} = 3$.

48. Let the middle two roots be $\alpha$ and $\beta$. Then the first and the last roots are $\frac{\alpha\beta}{2\alpha-\beta}$ and $\frac{\alpha\beta}{2\beta-\alpha}$. We have $\frac{1}{40} = (\frac{\alpha\beta}{2\beta-\alpha})\alpha\beta(\frac{\alpha\beta}{2\alpha-\beta}) = \frac{\alpha^3\beta^3}{(2\beta-\alpha)(2\alpha-\beta)}$ and $-\frac{1}{20} = \frac{\alpha^2\beta^2}{2\beta-\alpha} + \frac{\alpha^2\beta^2}{2\alpha-\beta} + \frac{\alpha^3\beta^2}{(2\beta-\alpha)(2\alpha-\beta)} + \frac{\alpha^2\beta^3}{(2\beta-\alpha)(2\alpha-\beta)} = \frac{2\alpha^2\beta^2(\alpha+\beta)}{(2\beta-\alpha)(2\alpha-\beta)}$. Dividing the first equation by the second, we have $\frac{\alpha\beta}{\alpha+\beta} = -1$. Let $k = \alpha\beta$. Then $\alpha+\beta = -k$ and $(2\beta-\alpha)(2\alpha-\beta) = 9\alpha\beta - 2(\alpha+\beta)^2 = 9k - 2k^2$. From $\frac{1}{40} = \frac{k^3}{9k-2k^2} = \frac{k^2}{9-2k}$, we have $0 = 40k^2 + 2k - 9 = (20k-9)(2k+1)$. From $\alpha\beta = k = -\frac{1}{2}$, we have $\alpha+\beta = \frac{1}{2}$ so that $\alpha$ and $\beta$ are the roots of $x^2-\frac{1}{2}x-\frac{1}{2} = 0$. Hence $0 = 2x^2-x-1 = (2x+1)(x-1)$ so that $\alpha = 1$ and $\beta = -\frac{1}{2}$. It follows that $\frac{\alpha\beta}{2\beta-\alpha} = \frac{1}{4}$ and $\frac{\alpha\beta}{2\alpha-\beta} = -\frac{1}{5}$. From $\alpha\beta = k = \frac{9}{20}$, we have $\alpha+\beta = -\frac{9}{20}$ so that $\alpha$ and $\beta$ are the roots of $x^2+\frac{9}{20}x+\frac{9}{20} = 0$. However, $20x^2+9x+9$ is not a factor of $40x^4-22x^3-21x^2+2x+1$ and $k = \frac{9}{20}$ must be rejected.

50. Let $x = y-1$. Then

$$\begin{aligned}
0 &= (y-1)^4 + 4(y-1)^3 + 2(y-1)^2 - 4(y-1) - 2 \\
&= y^4 - 4y^3 + 6y^2 - 4y + 1 + 4y^3 - 12y^2 + 12y - 4 + 2y^2 - 4y + 2 \\
&\quad -4y + 4 - 2 \\
&= y^4 - 4y^2 + 1.
\end{aligned}$$

52. Let $x = y+2$. Then

$$\begin{aligned}
0 &= (y+2)^6 - 12(y+2)^5 + 3(y+2)^2 - 17(y+2) + 300 \\
&= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64 - 12y^5 - 120y^4 \\
&\quad -480y^3 - 960y^2 - 960y - 384 + 3y^2 + 12y + 12 - 17y - 34 + 300 \\
&= y^6 - 60y^4 - 320y^3 - 717y^2 - 773y - 42.
\end{aligned}$$

54. Let $y = x-3$ so that $x = y+3$. We have

$$\begin{aligned}
0 &= (y+3)^5 - 4(y+3)^4 + 3(y+3)^2 - 4(y+3) + 6 \\
&= y^5 + 15y^4 + 90y^3 + 270y^2 + 405y + 243 - 4y^4 - 48y^3 - 216y^2 - 432y - 324 \\
&\quad +3y^2 + 18y + 27 - 4y - 12 + 6 \\
&= y^5 + 11y^4 + 42y^3 + 57y^2 - 13y - 60.
\end{aligned}$$

56. Let $y = x^2$ so that $x = \sqrt{y}$. We have $(\sqrt{y})^4+(\sqrt{y})^3+2(\sqrt{y})^2+\sqrt{y}+1 = 0$. This is equivalent to $(y+1)^2 = -\sqrt{y}(y+1)$. Squaring yields $(y+1)^4 = y(y+1)^2$ or

$$0 = (y^2+2y+1)(y^2+y+1) = y^4 + 3y^3 + 4y^2 + 3y + 1.$$

58. Let $y = \frac{k}{x}$ so that $x = \frac{k}{y}$. Then the desired equation is $(\frac{k}{y})^3 + q(\frac{k}{y}) + r = 0$ or $ry^3 + kqy^2 + k^3 = 0$.

60. Let $k = -1$ in Problem 20. Then the desired equation is $ry^3 - qy^2 - 1 = 0$.

62. Let $y = -x^2$ so that $x = \sqrt{y}i$. Then the desired equation is $(\sqrt{y}i)^3 + q(\sqrt{y}i) + r = 0$ or $r = \sqrt{y}i(y - q)$. Squaring yields $r^2 = -y(y^2 - 2qy + q^2)$ so that $y^3 - 2qy^2 + q^2y + r^2 = 0$.

64. First, note that $\frac{\beta}{\gamma} + \frac{\gamma}{\beta} = \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2+2\beta\gamma}{\beta\gamma} - 2 = \frac{\alpha(\beta+\gamma)^2}{\alpha\beta\gamma} - 2 = \frac{\alpha^3}{-r^2} - 2 = \frac{-\alpha^3-2r}{r}$. Let $y = \frac{-x^3-2r}{r}$ so that $x = -\sqrt[3]{r(y+2)}$. Then the desired equation is $-r(y+2) - q\sqrt[3]{r(y+2)} + r = 0$ or $-r(y+1) = q\sqrt[3]{r(y+2)}$. Cubing yields $-r^3(y^3 + 3y^2 + 3y + 1) = q^3r(y+2)$. This is equivalent to $r^3y^3 + 3r^3y^2 + (3r^2 + q^3)y + r(r^2 + 2q^3) = 0$.

66. We have $5 = \alpha + (-\alpha) + \beta + (-\beta) + \gamma = \gamma$ and $20 = (-\alpha^2)(-\beta^2)\gamma = 5\alpha^2\beta^2$ so that $\alpha\beta = 2$. Now $-5 = \alpha(-\alpha) + \beta(-\beta) + \alpha\beta + \alpha(-\beta) + (-\alpha)\beta + (-\alpha)(-\beta) + \alpha\gamma + (-\alpha)\gamma + \beta\gamma + (-\beta)\gamma = -\alpha^2 - \beta^2$. Hence $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta = 5 + 4 = 9$ so that $\alpha + \beta = 3$. Along with $\alpha\beta = 2$, we see that $\alpha$ and $\beta$ are the roots of $0 = x^2 - 3x + 2 = (x-1)(x-2)$. Hence $\alpha = 1$ and $\beta = 2$. The five roots are $\pm 1$, $\pm 2$ and 5.

### Answers to Odd-numbered Exercises IV

1. $6x^4 - 13x^3 - 12x^2 + 39x - 18 = 0$.

3. $x^6 - 5x^5 - 8x^4 + 40x^3 + 16x^2 - 80x = 0$.

5. 1,3,5,7.
7. $\frac{1}{2}, \frac{1}{2}, -6$.
9. $-\frac{3}{2}, -2, 4$

11. $\pm\sqrt{3}, \frac{3}{4}, -\frac{1}{2}$.
13. $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$.
15. $-4, -1, 2, 5$.

17. $-\frac{4}{3}, -\frac{3}{2}, -\frac{5}{3}$.
19. (a) $-6q$; (b) $\frac{q}{r}$.
21. $2q^2$.

23. $3, -\frac{2}{3}, \frac{1\pm\sqrt{3}i}{2}$.
25. $-1 \pm \sqrt{2}, -1 \pm i$.

27. $-1, \pm\sqrt{3}, 1 \pm 2i$.
29. $x^4 - 8x^2 + 36 = 0$.

31. $x^4 - 10x^2 + 1 = 0$.
33. $x^4 - 6x^3 + 18x^2 - 26x + 21 = 0$.

35. One positive, one negative, two imaginary.

37. At least one positive, at least one negative, at least four imaginary.

39. $y^3 - 24y^2 + 9y - 24 = 0.$

41. $1, 1, -2, -\frac{1}{2}.$

43. $1, \frac{1\pm\sqrt{3}i}{2}, \frac{3\pm\sqrt{5}}{2}.$

45. $4, 2, \frac{4}{3}.$

49. $y^3 - 2y + 1 = 0.$

51. $y^5 - 7y^3 + 12y^2 - 7y = 0.$

53. $y^3 - \frac{9}{2}y^2 + \frac{13}{2}y - \frac{15}{4} = 0.$

55. $y^3 - 8y^2 + 19y - 15 = 0.$

57. $y^3 + 33y^2 + 12y + 8 = 0$

59. $y^3 - q^2y^2 - 2qr^2y - r^4 = 0.$

61. $ry^3 + q(1-r)y^2 + (1-r)^3 = 0.$

63. $y^3 + 3ry^2 + (q^3 + 3r^2)y + r^3 = 0.$

# INTERLUDE

Some historical accounts are presented to mark the midpoint of the text and to highlight the significance of the following chapters on the solution of the quartic and cubic equations. We start with Vieta with whose work on coefficients and roots we have become familiar. The stories give a picture of the mathematics of the Italian and French Renaissance.

## Vieta

Francois Vieta was born at Fontenay near la Rochelle, and died in Paris in 1603. Professionally a lawyer, member of Parliament, and public servant, Vieta was a firm believer in the divine right of kings and probably a zealous catholic. He gave up most of his leisure time to mathematics and is credited with the invention of symbolic algebra.

Vieta's mathematical reputation reached his King Henry IV who asked him to decipher a Spanish military code of some 600 characters. Vieta succeeded in breaking the code and for two years the French used it to their great profit. So convinced was Phillip II that the code could not be broken that he accused the French of using sorcery against him, "contrary to the practice of Christian faith.".

Among his researches into algebra and geometry Vieta is remembered for one of the first attempts to approximate $\pi$ by means of infinite series.

## Tartaglia

Niccolo Fontana, generally known as Tartaglia, the stammerer, was born at Brescia in 1500 and died at Venice on December 14, 1557. After the capture of the town by the French 1512, most of the inhabitants took refuge in the cathedral, and were there massacred by the soldiers. His father, a postal messenger, was amongst the killed. The boy himself had his skull split through in three places, while his jaws and his palate were cut open; he was left for dead, but his mother got into the cathedral, and finding him alive managed to carry him off. Deprived of all resources she recollected that dogs when wounded always licked the injured place, and to that remedy he attributed his ultimate recovery. The injury to his palate produced an impediment in his speech, from which he received his nickname.

His mother managed to get together sufficient money to pay for his attendance at school for fifteen days, and he took advantage of it to steal a copy-book from which he subsequently taught himself to read and write; but so poor were they that he tells us they could not afford paper, and was obliged to make use of the tombstones as slates to work his exercises.

He was appointed to the chair of mathematics at Venice and became famous through his acceptance of a challenge from Fiore. Fiore had learned from his master Ferro an empirical solution of a cubic equation of the form $x^3 + qx = r$.

This solution was previously unknown in Europe, and it is possible that that Ferro had found the result in an Arab work. Tartaglia, in answer to the challenge, stated that he could effect the solution of a numerical equation of the form $x^3 + px^2 = r$. Fiore, believing that Tartaglia was an impostor, challenged him to a contest. According to this challenge each of them was to deposit a certain stake with a notary, and whoever could solve the most problems out of a collection of thirty propounded by the other was to get the stakes, thirty days being allowed.

Tartaglia was aware that his adversary was acquainted with the solution of a cubic equation of some particular form, and suspecting that the questions proposed to him would all depend on the solution of such cubic equations, set himself the problem to find a general solution, and certainly discovered how to solve some if not all cubic equations. His solution is believed to have depended on a geometrical construction, but led to the formula which is often, but unjustly, described as Cardan's.

When the contest took place, all the questions proposed to Tartaglia were, as he had suspected, reducible to the solution of a cubic equation, and he succeeded in two hours in bringing them to particular cases of the equation $x^3 + qx = r$, of which he knew the solution. His opponent failed to solve any of the problems proposed to him, most of which were, as a matter of fact, reducible to the form $x^3 + px^2 = r$.

Tartaglia imitated Pacioli by inserting a large collection of mathematical puzzles in his publications. He half apologizes for them by explaining that it was not uncommon at dessert to propose arithmetical questions to company by way of amusement.

### Cardan

The life of Tartaglia was embittered by a quarrel with his contempary Cardan, who published Tartaglia's solution to the cubic equation which he obtained under a pledge of secrecy. Cardan was born at Pavia in 1501 and died in Rome in 1576. His career is an account of the most extraordinary and inconsistent acts. A gambler, if not a murderer, he was also an ardent student of science, solving problems which had long baffled all investigation; at one time of his life he was devoted to intrigues which were a scandal even in the sixteenth century, at another he did nothing but rave on astrology, and yet at another he declared that philosophy was the only subject worth of a man's attention. His was the genius that was closely allied with madness.

The illegitimate son of a lawyer of Milan, he was educated at the universities of Pavia and Padua. After taking his degree he commenced life a a doctor and it was at this time that he published his chief works. After spending a year or so in France, Scotland, and England he was elected to a chair at Pavia. Here he divided his time between debauchery, astrology, and mechanics. His two sons were as wicked and passionate as himself: the elder was executed for poisoning his wife, and about the same time Cardan in a fit of rage cut off the ears of the younger who had committed some offense; for this scandalous outrage he suffered no punishment as pope Gregory XIII granted him protection.

In 1562 Cardan moved to Bologna, but the scandals connected with his name were so great that the university took steps to prevent his lecturing, and only gave way under pressure from Rome. Later he was imprisoned for heresy on account of his having published the horoscope of Christ, and when released he found himself so generally detested that he resigned his chair. He left Bologna for Rome and as the most distinguished astrologer of his time received a pension as astrologer to the papal court. This proved fatal to him for, having fortold that he should die on a particular day, he felt obliged to commit suicide in order to keep up his reputation – so the story goes.

Cardan's chief mathematical work is his Ars Magna, 1545. Cardan was much interested in the contest between Tartaglia and Fiore, and as he had already begun writing this book he asked Tartaglia to communicate his method of solving the cubic. Tartaglia refused, whereupon Cardan abused him in the most violent terms, but shortly afterward wrote a note saying that a certain Italian nobleman had heard of Tartaglia's fame and was most anxious to meet him. Tartaglia came to Milan, and though he found no nobleman awaiting him at the end of his journey, he yielded to Cardan's importunity, and gave him the rule, Cardan on his side taking a solemn oath not to reveal it. Cardan asserts that he was given merely the result, and that he obtained the proof himself, but this is doubtful. He seems to have at once taught the method, and one of his pupils Ferrari reduced the quartic to a cubic so solved it.

When the Ars Magna was published the breach of faith was made manifest. Not unnaturally, Tartaglia was very angry, and after an acrimonious controversy sent a challenge to Cardan to take part in a mathematical duel. The preliminaries were settled, and the place of meeting was to be a certain church in Milan, but when the day arrived Cardan failed to appear, and sent Ferarri in his stead. Both sides claimed the victory, though it would appear that Tartaglia was the more successful; his opponents having broken up the meeting and he once again counting himself lucky to have escaped another church with his life. Not only did Cardan succeed in his fraud, but modern writers (including Hall and Knight) have often attributed the solution to him.

–from *The History of Mathematics* by W.W.R.Ball.

# CHAPTER V — QUARTIC EQUATIONS

In Chapters I and II we thoroughly treated quadratic equations. Then Chapters III and IV gave some insight into higher order equations. It is natural to ask, however, whether there is a general solution similar to the quadratic formula for the cubic or quartic. We look first at the quartic with the idea of separating it into two quadratics.

We shall now give a brief discussion of some of the methods which are employed to obtain the general solution of a quartic equation. It will be found that in each of the methods we have first to solve an auxiliary cubic equation; and thus it will be seen that as in the case of the cubic (next section), the general solution is not adapted for writing down.

### Descartes' Method

The following solution was given by Descartes in 1637. Suppose that the quartic equation is reduced to the form $x^4 + qx^2 + rx + s = 0$. Assume that it can be factored as $(x^2 + kx + m)(x^2 - kx + n)$. Then by equating coefficients, we have $m + n - k^2 = q$, $k(n - m) = r$ and $mn = s$. From the first two of these equations, we obtain $2n = k^2 + q + \frac{r}{k}$ and $m = k^2 + q - \frac{r}{k}$. Hence substituting in the third equation, $(k^3 + qk + r)(k^3 + qk - r) = 4sk^2$ or $k^6 + 2qk^4 + (q^2 - 4s)k^2 - r^2 = 0$.

This is a cubic in $k^2$ which always has one real positive solution. Thus when $k^2$ is known the values of $m$ and $n$ are determined, and the solution of the quartic is obtained by solving the two quadratics $x^2 + kx + m = 0$ and $x^2 - kx + n = 0$.

**Example 1.**
Solve the equation $x^4 - 2x^2 + 8x - 3 = 0$.

**Solution:**

Assume that $x^4 - 2x^2 + 8x - 3 = (x^2 + kx + m)(x^2 - kx + n)$. Then by equating coefficients, we have $m + n - k^2 = -2$, $k(n - m) = 8$ and $mn = -3$. Thus we obtain $(k^3 - 2k + 8)(k^3 - 2k - 8) = -12k^2$, or $k^6 - 4k^4 + 16k^2 - 64 = 0$. This equation is satisfied when $k^2 - 4 = 0$, or $k = \pm 2$. It will be sufficient to consider one of the values of $k$. Putting $k = 2$, we have $m + n = 2$ and $n - m = 4$, that is, $m = -1$ and $n = 3$. Thus $x^4 - 2x^2 + 8x - 3 = (x^2 + 2x - 1)(x^2 - 2x + 3)$. Hence $x^2 + 2x - 1 = 0$ or $x^2 - 2x + 3 = 0$, and therefore the roots are $-1 \pm \sqrt{2}$ and $1 \pm \sqrt{2}i$.

### Ferrari's Method

The solution of a quartic equation was first obtained by Ferrari, a pupil of Cardan, as follows. Denote the equation by $x^4 + 2px^3 + qx^2 + 2rx + s = 0$. Add to each side $(ax + b)^2$, the quantities $a$ and $b$ being determined so as to make the left side a perfect square. Then

$$x^4 + 2px^2 + (q + a^2)x^2 + 2(r + ab)x + (s + b^2) = (ax + b)^2.$$

Suppose that the left side of the equation is equal to $(x^2 + px + k)^2$. Then by comparing the coefficients, we have $p^2 + 2k = q + a^2$, $pk = r + ab$ and $k^2 = s + b^2$. By eliminating $a$ and $b$ from these equations, we obtain $(pk - r)^2 = (2k + p^2 - q)(k^2 - s)$ or $2k^3 - qk^2 + 2(pr - s)k - (p^2s - qs + r^2) = 0$.

From this cubic equation one real value of $k$ can always be found. Thus $a$ and $b$ are known. Also, $(x^2 + px + k)^2 = (ax + b)^2$. It follows that $x^2 + px + k = \pm(ax + b)$, and the values of $x$ are to be obtained from the two quadratics $x^2 + (y - a)x + (k - b) = 0$ and $x^2 + (p + a)x + (k + b) = 0$.

**Example 2.**
Solve the equation $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$.

**Solution:**

Add $a^2x^2 + 2abx + b^2$ to each side of the equation, and assume

$$x^4 - 2x^3 + (a^2 - 5)x^2 + 2(ab + 5)x + (b^2 - 3) = (x^2 - x + k)^2.$$

Then by equating coefficients, we have $a^2 = 2k + 6$, $ab = -k - 5$ and $b^2 = k^2 + 3$. It follows that $(2k + 6)(k^2 + 3) = (k + 5)^2$ or $2k^3 + 5k^2 - 4k - 7 = 0$. By trial, we find that $k = -1$. Hence $a^2 = 4$, $b^2 = 4$ and $ab = -4$. From the assumption, it follows that $(x^2 - x + k)^2 = (ax + b)^2$. Substituting the values of $k$, $a$ and $b$, we have the two equations $x^2 - x - 1 = \pm(2x - 2)$, that is, $x^2 - 3x + 1 = 0$ and $x^2 + x - 3 = 0$. Thus the roots are $\frac{3 \pm \sqrt{5}}{2}$ and $\frac{-1 \pm \sqrt{13}}{2}$.

**Theorem - On Quintics and Higher Order Equations**
There is no general algebraical solution of equations of a degree higher than the fourth. The proof that this is impossible required modern methods and is a highlight of modern mathematics. If, however, the coefficients of an equation are numerical, the value of any real root may be approximated to any required degree of accuracy. Vieta discovered an algorithm, but the calculations involved for higher order equations were so lengthy that it was commented that they involved, "work not fit for a Christian." The commentators were right of course: we now let computers do the work.

## EXERCISES V

Solve the following equations. More problems appear in Chapter VI.

1. $x^4 - 3x^2 - 42x - 40 = 0$.
2. $x^4 - 10x^2 - 20x - 16 = 0$.
3. $x^4 - 3x^2 - 6x - 2 = 0$.
4. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$
5. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$.
6. $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$.
7. $4x^4 - 20x^3 + 33x^2 - 20x + 4 = 0$.
8. $x^5 - 6x^4 - 17x^3 + 17x^2 + 6x - 1 = 0$.

## Solutions to Even-numbered Exercises V

2. Let $0 = x^4-10x^2-20x-16 = (x^2+kx+\ell)(x^2-kx+m) = x^4+(m+\ell-k^2)x^2+k(m-\ell)x+m\ell$. Then $m+\ell = k^2-10$, $m-\ell = -\frac{20}{k}$ and $m\ell = -16$. From the first two of these three equations, we have $2m = k^2-10-\frac{20}{k}$ and $2\ell = k^2-10+\frac{20}{k}$. Using the third equation, we have $-64 = 4m\ell = (k^2-10-\frac{20}{k})(k^2-10+\frac{20}{k}) = (k^2-10)^2-(\frac{20}{k})^2 = k^4-20k^2+100-\frac{400}{k^2}$. This is equivalent to $k^6-20k^4+164k^2-400 = 0$. By inspection, $k = 2$ is a root. Hence $m = -8$ and $\ell = 2$. From $x^2+2x+2 = 0$, we have $x = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm i$. From $0 = x^2-2x-8 = (x-4)(x+2)$, we have $x = 4$ or $-2$.

4. We have

$$\begin{aligned}(ax+b)^2 &= x^4+2x^3+(a^2-7)x^2+2(ab-4)x+(b^2+12)\\ &= (x^2+x+k)^2\\ &= x^4+2x^3+(2k+1)x^2+2kx+k^2.\end{aligned}$$

Hence $a^2 = 2k+8$, $ab = k+4$ and $b^2 = k^2-12$. It follows that

$$(k+4)^2 = (ab)^2 = a^2b^2 = (2k+8)(k^2-12).$$

This is equivalent to $2k^3+7k^2-32k-112 = 0$. By inspection, $k = 4$ is a root. Since $ab = 8$, $a$ and $b$ are of the same sign. Hence $a = 4$ and $b = 2$. From $x^2+x+4 = 4x+2$, we have $0 = x^2-3x+2 = (x-1)(x-2)$ so that $x = 1$ or 2. From $x^2+x+4 = -4x-2$, we have $0 = x^2+5x+6 = (x+2)(x+3)$ so that $x = -2$ or $-3$.

6. We have

$$\begin{aligned}(ax+b)^2 &= x^4-2x^3+(a^2-12)x^2+2(ab+5)x+(b^2+3)\\ &= (x^2-x+k)^2\\ &= x^4-2x^3+(2k+1)x^2-2kx+k^2.\end{aligned}$$

Hence $a^2 = 2k+13$, $ab = -k-5$ and $b^2 = k^2-3$. It follows that

$$(-k-5)^2 = (ab)^2 = a^2b^2 = (2k+13)(k^2-3).$$

This is equivalent to $k^3+6k^2-8k-32 = 0$. By inspection, $k = -2$ is a root. Since $ab = -3$, $a$ and $b$ are of opposite signs. Hence $a = 3$ and $b = -1$. From $x^2-x-2 = 3x-1$, we have $x^2-4x-1 = 0$ so that $x = \frac{4\pm\sqrt{16+4}}{2} = 2\pm\sqrt{5}$. From $x^2-x-2 = -3x+1$, we have $0 = x^2+2x-3 = (x-1)(x+3)$ so that $x = 1$ or $-3$.

8. We have

$$\begin{aligned}0 &= x^5-6x^4-17x^3+17x^2+6x-1\\ &= (x^5-1)-6x(x^3-1)-17x^2(x-1)\\ &= (x-1)(x^4+x^3+x^2+x^1-6(x^2+x+1)-17x^2)\\ &= (x-1)(x^4-5x^3-22x^2-5x+1).\end{aligned}$$

Hence one of the roots is 1. Let $y = x + \frac{1}{x}$. Then $y^2 = x^2 + \frac{1}{x^2} + 2$. Hence

$$0 = x^2 - 5x - 22 - \frac{5}{x} + \frac{1}{x^2} = y^2 - 5y - 24 = (y - 8)(y + 3).$$

From $x + \frac{1}{x} = 8$, we have $x^2 - 8x + 1 = 0$ so that $x = \frac{8 \pm \sqrt{64-4}}{2} = 4 \pm \sqrt{15}$.
From $x + \frac{1}{x} = -3$, we have $x^2 + 3x + 1 = 0$ so that $x = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$.

## Answers to Odd-numbered Exercises V

1. $4, -1, \frac{-3 \pm \sqrt{31}i}{2}$. 3. $1 \pm \sqrt{2}, -1 \pm i$. 5. $\pm 1, -4 \pm \sqrt{6}$. 7. $2, 2, -\frac{1}{2}, -\frac{1}{2}$.

7

# CHAPTER VI — CUBIC EQUATIONS

1. The Cube Roots of Unity
2. Cardan's Method

## The Cube Roots of Unity

Perhaps the most simple cubic equation is $x^3 - 1 = 0$. One root is clearly $x = 1$, but we uncover the other two by factoring. Hence, $(x-1)(x^2+x+1) = 0$. Therefore, either $x - 1 = 0$, or $x^2 + x + 1 = 0$; whence $x = 1$ or $x = \frac{-1\pm\sqrt{3}i}{2}$.

It may be shown by actual computation that each of these values when cubed is equal to unity (try it). Thus unity has three cube roots, 1, $\frac{-1+\sqrt{3}i}{2}$ and $\frac{-1-\sqrt{3}i}{2}$, two of which are complex numbers.

Let us denote these by $\alpha$ and $\beta$; then since they are the roots of the equation $x^2 + x + 1 = 0$, their product is equal to unity. We have then $\alpha\beta = 1$. Since $\alpha^3 = 1, \beta = \alpha^3\beta = \alpha^2$. Similarly we may show that $\alpha = \beta^2$.

Since each of the imaginary roots is the square of the other, it is usual to denote the three cube roots of unity by 1, $\omega$, and $\omega^2$.

Also $\omega$ satisfies the equation $x^2 + x + 1 = 0$; so $\omega^2 + \omega + 1 = 0$. In other words, *the sum of the three cube roots of unity is zero*. Again, $\omega \cdot \omega^2 = \omega^3 = 1$. Therefore,

(1) the product of the two imaginary roots is unity;

(2) every integral power of $\omega^3$ is unity.

It is useful to notice that the successive positive integral powers of $\omega$ are 1, $\omega$ and $\omega^2$; for, if $n$ be a multiple of 3, it must be of the form $3m$, and $\omega^n = \omega^{3m} = 1$. Conversely, if $n$ be not a multiple of 3, it must be of the form $3m + 1$ or $3m + 2$. If $n = 3m + 1$, $\omega^n = \omega^{3m+1} = \omega^{3m} \cdot \omega = \omega$. If $n = 3m + 2$, $\omega^n = \omega^{3m+2} = \omega^{3m} \cdot \omega^2 = \omega^2$.

We now see that every real number has three cube roots, two of which are complex. For the cube roots of $a^3$ are those of $a^3 \cdot 1$, and therefore are $a$, $a\omega$ and $a\omega^2$. Similarly the cube roots of 9 are $\sqrt[3]{9}$, $\omega\sqrt[3]{9}$ and $\omega^2\sqrt[3]{9}$, where $\sqrt[3]{9}$ is the cube root found by the ordinary arithmetical rule. In future, unless otherwise stated, the symbol $\sqrt[3]{a}$ will always be taken to denote the arithmetical cube root of $a$.

**Example 1.**
Resolve $x^3 + y^3$ into three factors of the first degree.

**Solution:**

Since $\omega + \omega^2 = -1$ and $\omega^3 = 1$, we have

$$\begin{aligned} x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\ &= (x+y)(x+\omega y)(x+\omega^2 y). \end{aligned}$$

**Example 2.**
Show that $(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) = a^2 + b^2 + c^2 - bc - ca - ab.$

**Solution:**

In the product of $a + \omega b + \omega^2 c$ and $a + \omega^2 b + \omega$ the coefficients of $b^2$ and $c^2$ are $\omega^3 = 1$; the coefficient of $bc$ is $\omega^2 + \omega^4 = \omega^2 + \omega = -1$; and the coefficients of $ca$ and $ab$ are $\omega^2 + \omega = -1$.

**Example 3.**
Show that $(1 + \omega - \omega^2)^3 - (1 - \omega + \omega^2)^3 = 0.$

**Solution:**

Since $1 + \omega + \omega^2 = 0$ we have

$$\begin{aligned} (1+\omega-\omega^2)^3 - (1-\omega+\omega^2)^3 &= (-2\omega^2)^3 - (-2\omega)^3 \\ &= -8\omega^6 + 8\omega^3 \\ &= -8 + 8 \\ &= 0. \end{aligned}$$

## Cardan's Method

The general type of a cubic equation is $x^3 + Px^2 + Qx + R = 0$, but as explained in Chapter IV, this equation can be reduced to the simpler form $x^3 + qx + r = 0$, which we shall take as the standard form of a cubic equation.

To solve the equation $x^3 + qx + r = 0$. Let $x = y + z$. Then $x^3 = y^3 + z^3 + 3yz(y + z) = y^3 + z^3 + 3yzx$, and the given equation becomes $y^3 + z^3 + (3yz + q)x + r = 0$.

At present $y$ and $z$ are any two quantities subject to the condition that their sum is equal to one of the roots of the given equation. If we further suppose that they satisfy the equation $3yz + q = 0$, they are completely determinate. We thus obtain $y^3 + z^3 = -r$ and $y^3z^3 = -\frac{q^2}{27}$. Hence $y^3$ and $z^3$ are the roots of the quadratic $t^2 + rt - \frac{q^2}{27} = 0$. Solving this equation, and putting $y^3 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$ and $z^3 = -\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}$, we obtain the value of $x$ from the relation $x = y + z$. Thus

$$x = \left(-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right)^{\frac{1}{3}}.$$

Each of $y^3$ and $z^3$ has three cube roots. Hence it would appear that $x$ has *nine* values. This, however is not the case. Since $yz = -\frac{q}{3}$, the cube roots are to be taken in pairs so that the product of each pair is rational.

Hence if $y$ and $z$ denote the values of any pair of cube roots which fulfill this condition, the only other admissible pairs will be $\omega y$ and $\omega^2 z$, as well as $\omega^2 y$ and $\omega z$, where $\omega$ and $\omega^2$ are the imaginary cube roots of unity. Hence the roots of the equation are $y + z$, $\omega y + \omega^2 z$ and $\omega^2 y + \omega z$.

**Example 1.**
Solve the equation $x^3 - 15x = 126$.

**Solution:**

Let $x = y + z$. Then $y^3 + z^3 + (3yz - 15)x = 126$. Let $3yz - 15 = 0$. Then $y^3 + z^3 = 126$ and $y^3z^3 = 125$. Hence $y^3$ and $z^3$ are the roots of the equation $0 = t^2 - 126t + 125 = (t - 125)(t - 1)$. It follows that $y^3 = 125$ and $z^3 = 1$, so that $y = 5$ and $z = 1$. Thus the roots are $y + z = 5 + 1 = 6$, $\omega y + \omega^2 z = \frac{-5+5\sqrt{3}i}{2} + \frac{-1-\sqrt{3}i}{2} = -3 + 2\sqrt{3}i$ and $\omega^2 y + \omega z = -3 - 2\sqrt{3}i$.

To explain the reason why we apparently obtain nine values for $x$ earlier, we observe that $y$ and $z$ are to be found from the equations $y^3 + z^3 = -r$ and $yz = -\frac{q}{3}$. However, in the process of solution, the second of these was changed into $y^3z^3 = -\frac{q^3}{27}$, which would also hold if $yz = -\frac{\omega q}{3}$ or $yz = -\frac{\omega^2 q}{3}$. Hence the other six values of $x$ are solutions of the cubics $x^3 + \omega qx + r = 0$ and $x^3 + \omega^2 qx + r = 0$.

We proceed to consider more fully the roots of the equation $x^3 + qx + r = 0$.

(i) If $\frac{r^2}{4} + \frac{q^3}{27}$ is positive, then $y^3$ and $z^3$ are both real. Let $y$ and $z$ represent their arithmetical cube roots. Then the roots are $y + z$, $\omega y + \omega^2 z$ and $\omega^2 y + \omega z$. The first of these is real, and by substituting for $\omega$ and $\omega^2$ the other two become $-\frac{y+z}{2} + \frac{(y-z)\sqrt{3}i}{2}$ and $-\frac{y+z}{2} - \frac{(y-z)\sqrt{3}i}{2}$.

(ii) If $\frac{r^2}{4} + \frac{q^3}{27}$ is zero, then $y^3 = z^3$. In this case $y = z$, and the roots become $2y$, $y(\omega + \omega^2) = -y$ and $y(\omega^2 + \omega) = -y$.

(iii) If $\frac{r^2}{4} + \frac{q^3}{27}$ is negative, then $y^3$ and $z^3$ are imaginary expressions of the form $a + bi$ and $a - bi$. Suppose that the cube roots of these quantities are $m + ni$ and $m - ni$. Then the roots of the cubic become $(m + ni) + (m - ni) = 2m$, $(m + ni)\omega + (m - ni)\omega^2 = -m - n\sqrt{3}$ and $(m + ni)\omega^2 + (m - ni)\omega = -m + n\sqrt{3}$, which are all real quantities. As however there is no general arithmetical or algebraical method of finding the exact value of the cube root of imaginary quantities, the solution obtained earlier is of little practical use when the roots of the cubic are all real and unequal.

This case is sometimes called the *Irreducible Case* of Cardan's solution. It may be completed by the use of trigonometry.

## EXERCISES VI

If 1, $\omega$ and $\omega^2$ are the three cube roots of unity, prove that:

1. $(1-\omega+\omega^2)(1+\omega-\omega^2)=4.$

2. $(1+\omega^2)^4=\omega.$

3. $(2+5\omega+2\omega^2)^6=(2+2\omega+5\omega^2)^6=729.$

4. $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5)=9.$

5. Prove that $x^3+y^3+z^3-3xyz=(x+y+z)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega).$

6. $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)\cdots(1-\omega^n+\omega^{2n})=2^{2n}.$

7. If $ax+cy+bz=X$, $cx+by+az=Y$, $bx+ay+cz=Z$, show that

$$(a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy)$$

$$=X^2+Y^2+Z^2-YZ-XZ-XY.$$

8. If $x=a+b$, $y=a\omega+b\omega^2$, $z=a\omega^2+b\omega$, show that

   (a) $xyz=a^3+b^3;$

   (b) $x^2+y^2+z^2=6ab;$

   (c) $x^3+y^3+z^3=3(a^3+b^3).$

Solve the following equations:

9. $x^3-18x=35.$

10. $x^3+72x-1720=0.$

11. $x^3+63x-316=0.$

12. $x^3+21x+342=0.$

13. $28x^3-9x^2+1=0.$

14. $x^3-15x^2-33x+847=0.$

15. $2x^3+3x^2+3x+1=0.$

16. Prove that the real root of the equation $x^3+12x-12=0$ is $2(\sqrt[3]{2})-\sqrt[3]{4}.$

Having now completed some standard exercises on quartics and cubics the student may solve the following problems using the strategies of Chapters IV, V, and VI.

17. Solve $x^4 + 9x^3 + 12x^2 - 80x - 192 = 0$, which has equal roots.

18. (a) Find the relation between $q$ and $r$ in order that the equation $x^3 + qx + r = 0$ may be put into the form $x^4 = (x^2 + ax + b)^2$.
    (b) Solve the equation $8x^3 - 36x + 27 = 0$.

19. Consider the equations $x^3 + 3px^2 + 3qx + r$ and $x^2 + 2px + q$.
    (a) If they have a common factor, show that $4(p^2 - q)(q^2 - pr) - (pq - r)^2 = 0$.
    (b) If they have two common factors, show that $p^2 - q = 0$ and $q^2 - pr = 0$.

20. If the equation $ax^3 + 3bx^2 + 3cx + d = 0$ has two equal roots, show that each of them is equal to $\frac{bc-ad}{2(ac-b^2)}$.

21. Show that the equation $x^4 + px^3 + qx^2 + rx + s = 0$ may be solved as a quadratic if $r^2 = p^2 s$.

22. Solve the equation $x^6 - 18x^4 + 16x^3 + 28x^2 - 32x + 8 = 0$, one of whose roots is $\sqrt{6} - 2$.

23. If $\alpha$, $\beta$, $\gamma$ and $\delta$ are the roots of the equation $x^4 + qx^2 + rx + s = 0$, find the equation whose roots are $\beta + \gamma + \delta + (\beta\gamma\delta)^{-1}$, $\gamma + \delta + \alpha_{(}\gamma\delta\alpha)^{-1}$, $\delta + \alpha + \beta + (\delta\alpha\beta)^{-1}$ and $\alpha + \beta + \gamma + (\alpha\beta\gamma)^{-1}$.

24. Consider the equation $x^4 - px^3 + qx^2 - rx + s = 0$.
    (a) Prove that $p^3 - 4pq + 8r = 0$ if the sum of two of the roots is equal to the sum of the other two roots.
    (b) Prove that $r^2 = p^2 s$ if the product of two of the roots is equal to the product of the other two roots.

25. Find the two roots of $x^5 - 209x + 56 = 0$ whose product is 1.

26. Find the two roots of $x^5 - 409x + 285 = 0$ whose sum is 5.

27. If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1}x + p_n = 0$, show that $(1 + \alpha_1^2)(1 + \alpha_2^2) \cdots (1 + \alpha_n^2) = (1 - p_2 + p_4 - \cdots)^2 + (p_1 - p_3 + p_5 - \cdots)^2$.

28. The sum of two roots of the equation $x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$ is 4. Solve the equation from the knowledge of this fact.

## Solutions to Even-numbered Exercises VI

2. We have $(1+\omega^2)^4 = (-\omega)^4 = \omega$ using $1+\omega+\omega^2 = 0$ and $\omega^3 = 1$.

4. Using $\omega^3 = 1$ and $1+\omega+\omega^2 = 0$, we have

$$\begin{aligned} & (1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^5) \\ = \; & (1-\omega)^2(1-\omega^2)^2 = (1-2\omega+\omega^2)(1-2\omega^2+\omega) \\ = \; & (-3\omega)(-3\omega^2) \\ = \; & 9. \end{aligned}$$

6. The product of the first two factors is $(1-\omega+\omega^2)(1-\omega^2+\omega^4) = (-2\omega)(-2\omega^2) = 2^2$. The product of the next two factors is $(1-\omega^{+}\omega^2)(1-\omega^2+\omega)$ again. It follows that the product of the $2n$ factors is $2^{2n}$.

8. (a) We have $a^3+b^3 = (a+b)(a^2-ab+b^2)$, so that it is only necessary to prove that $yz = a^2-ab+b^2$. Indeed, we have

$$yz = a^2\omega^3+ab\omega^2+ba\omega^4+b^2\omega^3 = a^2+ab(\omega^2+\omega)+b^2 = a^2-ab+b^2.$$

   (b) We have $x^2 = a^2+2ab+b^2$, $y^2 = a^2\omega^2+2ab\omega^3+b^2\omega^4$ and $z^2 = a^2\omega^4+2ab\omega^3+b^2\omega^2$. Hence $x^2+y^2+z^2 = a^2(1+\omega^2+\omega)+ab(1+1+1)+b^2(1+\omega+\omega^2) = 3ab$.

   (c) We have $y^3 = a^3\omega^3+3a^2b\omega^4+3ab^2\omega^5+b^3\omega^6$, $z^3 = a^3\omega^6+3a^2b\omega^5+3ab^2\omega^4+b^3\omega^3$ and $x^3 = a^3+3a^2b+3ab^2+b^3$. Hence

$$\begin{aligned} x^3+y^3+z^3 &= a^3(1+1+1)+3a^2b(1+\omega+\omega^2) \\ &\quad +3ab^2(1+\omega^2+\omega)+b^3(1+1+1) \\ &= 3(a^3+b^3). \end{aligned}$$

10. We have $y^3+z^3 = 1720$ while $y^3z^3 = -\frac{72^3}{27} = -13824$. From

$$0 = t^2-1720t-13824 = (t-1728)(t+8),$$

we have $y = \sqrt[3]{1728} = 12$ and $z = \sqrt[3]{-8} = -2$. Hence the roots of the cubic equation are $y+z = 10$, $y\omega+z\omega^2 = -6+6\sqrt{3}i+1+\sqrt{3}i = -5+7\sqrt{3}i$ and $y\omega^2+z\omega = -5-7\sqrt{3}i$.

12. We have $y^3+z^3 = -342$ while $y^3z^3 = -\frac{21^3}{27} = -343$. From $0 = t^2+342t-343 = (t-1)(t+343)$, we have $y\sqrt[3]{1} = 1$ and $z = \sqrt[3]{-343} = -7$. Hence the roots of the cubic equation are $y+z = -6$, $y\omega+z\omega^2 = -\frac{1}{2}+\frac{\sqrt{3}}{2}i+\frac{7}{2}+\frac{7\sqrt{3}}{2}i = 3+4\sqrt{3}i$ and $y\omega^2+z\omega = 3-4\sqrt{3}i$.

14. Let $w = x - 5$ so that $x = w + 5$. Then

$$0 = (w+5)^3 - 15(w+5)^2 - 33(w+5) + 847 = w^3 - 108w + 432.$$

We have $y^3 + z^3 = -432$ while $y^3z^3 = \frac{108^3}{27} = 46656$. From $0 = t^2 + 432t - 46656 = (t+216)^2$, we have $y = z = \sqrt[3]{-216} = -6$. Hence the roots of the transformed cubic equation are $y + z = -12$, $y\omega + z\omega^2 = 3 - 3\sqrt{3}i + 3 + 3\sqrt{3}i = 6$ and $y\omega^2 + z\omega = 6$, so that the roots of the original cubic equation are $-7$, 11 and 11.

16. We have $y^3+z^3 = 12$ while $y^3z^3 = -\frac{12^3}{27} = -64$. From $0 = t^2-12t-64 = (t-16)(t+4)$, we have $y = \sqrt{316} = 2(\sqrt[3]{2})$ and $z = \sqrt[3]{-4} = -\sqrt[3]{4}$. Hence the real root of the cubic equation is $y + z = 2(\sqrt[3]{2}) - \sqrt[3]{4}$.

18. (a) Comparing $x^4 = (x^2+ax+b) = x^4 - +2ax^3 + (a^2+2b)x^2 + 2abx + b^2$ with $0 = 2ax^3 + 2aqx + 2ar$, we have $a^2 + b = 0$, $q = b$ nd $2ar = b^2$. Hence $q^4 = b^4 = 4a^2r^2 = 4(-2b)r^2 = -8qr^2$ so that $q^3 + 8r^2 = 0$.

    (b) In $8x^3-36x+27 = 0$, $q = -\frac{9}{2}$ while $r = \frac{27}{8}$, and indeed $q^3+8r^2 = 0$. Thus we may take $b = q = -\frac{9}{2}$ and $a = \frac{b^2}{2r} = 3$ so that $x^4 = (x^2 + 3x - \frac{9}{2})^2$. From $x^2 = x^2 + 3x - \frac{9}{2}$, we have $x = \frac{3}{2}$. From $-x^2 = x^2+3x-\frac{9}{2}$, we have $4x^2+6x-9 = 0$ so that $x = \frac{-6\pm\sqrt{36+144}}{8} = \frac{-3\pm3\sqrt{5}}{4}$.

20. Let the roots be $\alpha$, $\alpha$ and $\beta$. Then $2\alpha + \beta = -\frac{3b}{a}$, $\alpha^2 + 2\alpha\beta = \frac{3c}{a}$ and $\alpha^2\beta = -\frac{d}{a}$. Now $\frac{9}{a^2}(bc-ad) = \frac{3b}{a}\cdot\frac{3c}{a} - 9\frac{d}{a} = (-2\alpha-\beta)(\alpha^2+2\alpha\beta)+9\alpha^2\beta = 2\alpha(-\alpha^2 + 2\alpha\beta - \beta^2)$ while $\frac{18}{a^2}(ac - b^2) = 6\frac{3c}{a} - 2(-\frac{3b}{a})^2 = 6(\alpha^2 + 2\alpha\beta) - 2(2\alpha + \beta)^2 = 2(-\alpha^2 + 2\alpha\beta - \beta^2)$. Division yields $\alpha = \frac{bc-ad}{2(ac-b^2)}$.

22. Since all coefficients are rational, irrational roots appear in conjugate pairs, so that $-2 - \sqrt{6}$ is also a root. Hence the original polynomial is divisible by

$$(x - (-2 + \sqrt{6}))(x - (-2 - \sqrt{6})) = (x+2)^2 - (\sqrt{6})^2 = x^2 + 4x - 2.$$

The quotient turns out to be $x^4 - 4x^3 + 8x - 4$. now

$$\begin{aligned}(ax+b)^2 &= x^4 - 4x^3 + a^2x^2 + 2(ab+4)x + (b^2-4)\\ &= (x^2 - 2x + k)^2\\ &= x^4 - 4x^3 + (2k+4)x^2 - 4kx + k^2.\end{aligned}$$

Hence $a^2 = 2k + 4$, $ab = -2k - 4$ and $b^2 = k^2 + 4$. It follows that

$$(-2k-4)^2 = (ab)^2 = a^2b^2 = (2k+4)(k^2+4).$$

This is equivalent to $0 = k^3 - 4k^2 = k(k-2)(k+2)$. We take the root $k = 0$. Since $ab = -4$, $a$ and $b$ are of opposite signs. Hence $a = 2$ and $b = -2$. From $x^2 - 2x = 2x - 2$, we have $x^2 - 4x + 2 = 0$ so that $x = \frac{4\pm\sqrt{16-8}}{2} = 2 \pm \sqrt{2}$. From $x^2 - 2x = -2x + 2$, we have $x^2 = 2$ so that $x = \pm\sqrt{2}$.

24. Let the roots be $\alpha$, $\beta$, $\gamma$ and $\delta$. Then $\alpha+\beta+\gamma+\delta = p$, $\alpha\beta\gamma\delta = s$, $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta = q$ and $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta = r$.

   (a) Suppose $\alpha+\beta = \gamma+\delta = \frac{p}{2}$. Then $\alpha\beta+\gamma\delta = \alpha\beta\frac{2(\gamma+\delta)}{p}+\gamma\delta\frac{2(\alpha+\beta)}{p} = \frac{2r}{p}$. Also, $\alpha\beta+\gamma\delta = q-\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta = q-\alpha(\gamma+\delta)-\beta(\gamma+\delta) = q-(\alpha+\beta)(\gamma+\delta) = q-\frac{p^2}{4}$. Hence $\frac{2r}{p} = q-\frac{p^2}{4}$ or $p^3-4pq+8r = 0$.

   (b) Suppose $\alpha\beta = \gamma\delta = \sqrt{s}$ instead. Then $\frac{r}{\sqrt{s}} = \frac{\alpha\beta\gamma+\alpha\beta\delta}{\alpha\beta}+\frac{\alpha\gamma\delta+\beta\gamma\delta}{\gamma\delta} = \gamma+\delta+\alpha+\beta = p$ so that $p^2s = r^2$.

26. Let the product of these two roots be $k$. Then the original polynomial is divisible by $x^2-5x+k$.

$$
\begin{array}{rrrrrrrr}
 & & & & x^3 & +5x^2 & +(25-k)x & +5(25-2k) \\
\cline{3-8}
x^2-5x+k & ) & x^5 & & & & -409x & +285 \\
 & & x^5 & -5x^4 & +kx^3 & & & \\
\cline{3-8}
 & & & 5x^4 & -kx^3 & & & \\
 & & & 5x^4 & -25x^3 & +5kx^2 & & \\
\cline{4-8}
 & & & & (25-k)x^3 & -5kx & -409x & \\
 & & & & (25-k)x^3 & -5(25-k)x & k(25-k)x & \\
\cline{5-8}
 & & & & & 5(25-2k)x^2 & -(409+25k-k^2)x & +285 \\
 & & & & & 5(25-2k)x^2 & -25(25-2k)x & +5(25-2k)k. \\
\cline{6-8}
\end{array}
$$

From $0 = k^2-75k+216 = (k-3)(k-72)$ and $0 = 10k^2-125k+285 = 5(k-3)(2k-19)$, we see that $k = 3$. From $x^2-5x+3 = 0$, we have $x = \frac{5\pm\sqrt{25-12}}{2} = \frac{5\pm\sqrt{13}}{2}$.

28. Let the product of these roots be $k$. Then the original polynomial is divisible by $x^2-4x+k$. Using Horner's method, we have

$$
\begin{array}{rrrrrrr}
 & & & & x^2 & -4x & -(k-5) \\
\cline{3-7}
x^2-4x+k & ) & x^4 & -8x^3 & +21x^2 & -20x & +5 \\
 & & x^4 & -4x^3 & +kx^2 & & \\
\cline{3-7}
 & & & -4x^3 & -(k-21)x^2 & -20x & \\
 & & & -4x^3 & +16x^2 & -4kx & \\
\cline{4-7}
 & & & & -(k-5)x^2 & +4(k-5)x & +5 \\
 & & & & -(k-5)x^2 & +4(k-5)x & -k(k-5) \\
\cline{5-7}
\end{array}
$$

From $0 = 5+k(k-5) = k^2-5k+5$, we have $k = \frac{5\pm\sqrt{25-20}}{2} = \frac{5\pm\sqrt{5}}{2}$. Taking $k = \frac{5+\sqrt{5}}{2}$, we have $x^2-4x+\frac{5+\sqrt{5}}{2} = 0$ so that $x = \frac{4\pm\sqrt{16-2(5+\sqrt{5})}}{2} = \frac{4\pm\sqrt{6-2\sqrt{5}}}{2} = \frac{4\pm(\sqrt{5}-1)}{2}$. This yields the roots $\frac{3+\sqrt{5}}{2}$ and $\frac{5-\sqrt{5}}{2}$. If we take $k = \frac{5-\sqrt{5}}{2}$ instead, we get the roots $\frac{3-\sqrt{5}}{2}$ and $\frac{5+\sqrt{5}}{2}$.

## Answers to Odd-numbered Exercises VI

9. $\frac{-5\pm\sqrt{3}i}{2}$.

11. $4, -2 \pm \sqrt{3}i$.

13. $-\frac{1}{4}, \frac{2\pm\sqrt{3}i}{2}$.

15. $-\frac{1}{2}, \frac{-1\pm\sqrt{3}i}{2}$.

17. $-4, -4, -4, 3$.

23. $s^3y^4 + qs(1-s)^2y^2 + r(1-s)^3y + (1-s)^4 = 0$.

25. $2 \pm \sqrt{3}$.

# CHAPTER VII — INEQUALITIES

1. Definition and Basic Theorems
2. Solving Polynomial Inequalities
3. AM–GM Inequality
4. The Values of Rational Functions

## Definition and Basic Theorems

**Definition** [$a > b$]
Any quantity $a$ is said to be greater than another quantity $b$ when $a - b$ is positive ; thus 2 is greater than $-3$, because $2 - (-3)$, or 5 is positive. Also $b$ is said to be less than $a$ when $b - a$ is negative; thus $-5$ is less than $-2$, because $-5 - (-2)$, or $-3$ is negative. In accordance with this definition, zero must be regarded as greater than any negative quantity.

**Theorem 1**
If $a > b$, $b > c$ *real* numbers, and $d > 0$, then it is evident that
(i) $a + d > b + d$;
(ii) $a - d > b - d$;
(iii) $ad > bd$;
(iv) $\frac{a}{d} > \frac{b}{d}$;
(v) $a > c$ (Transitivity).

That is, an inequality will still hold after each side has been increased, decreased, multiplied, or divided by the same positive quantity. Moreover there is a logical connection between certain statements having a quantity in common.

**Proof of (i):**

Suppose $a > b$, then $a - b$ is positive. Then $a - b + d - d$ is positive, and therefore $a + d - (b + d)$ is positive. By definition then, $a + d > b + d$. The proofs of (ii)–(iv) are similar and left to the reader.

**Proof of (v):**

Since $a > b$ and $b > c$ we have $a - b$ and $b - c$ both positive. Since both are positive, their sum is positive hence $(a - b) + (b - c) = a - c$ is positive. So $a > c$.

**Example 1**
Solve $5x - 20 < 40$.

**Solution**

By Theorem 1(i) we have $5x < 60$ and therefore, by Theorem 1(iv), $x < 12$.

**Example 2**
Suppose $y$ is positive, $x < y$, and $3y < z$, show that $x < z$.

**Solution:**

$3y - y = 2y > 0$ since $y$ is positive. Therefore $3y > y$. Then $x < y < 3y < z$ implies $x < z$ by transitivity.

**Notation**
The symbols $\geq$ and $\leq$ shall be used to include equality. We shall read $x \leq 3$ as '$x$ is less than or equal to 3'. The student must use discretion in determining whether this distinction is important in each context.

**Theorem 2**
If each side of an inequation is multiplied or divided by a negative quantity, the inequality must be reversed.

**Proof:**

Suppose $a > b$ and $d$ is positive. Then $a - b$ is positive and $b - a = (-a) - (-b)$ is negative, therefore $-a < -b$ and by (iii) above $-da < -db$. Hence $(-d)a < (-d)b$, therefore multiplication by $-d$ reverses the inequality. Similarly we have $\frac{a}{-d} < \frac{b}{-d}$, therefore division by $-d$ also reverses the inequality.

**Theorem 3**
If $a_1 > b_1,\ a_2 > b_2,\ a_3 > b_3,\ \cdots,\ a_m > b_m$, all positive, it is clear that

$$\begin{aligned} a_1 + a_2 + a_3 + \cdots + a_m &> b_1 + b_2 + b_3 + \cdots + b_m \ ; \\ a_1 a_2 a_3 \cdots a_m &> b_1 b_2 b_3 \cdots b_m. \end{aligned}$$

A particular case of the later is if

$$\begin{aligned} a_1 = a_2 = \cdots = a_m &= a, \\ b_1 = b_2 = \cdots = b_m &= b, \end{aligned}$$

then we have $a^m > b^m$ for any positive integer $m$.

**Proof:**

Starting with $a_1 > b_1$ and adding $a_2$ we have

$$a_1 + a_2 > b_1 + a_2.$$

Since $a_2 > b_2$, by adding $b_1$ we have

$$b_1 + a_2 > b_1 + b_2.$$

By transitivity, Theorem 1(v), we have $a_1 + a_2 > b_1 + b_2$. Continuing in this way one can extend the result to $m$ terms on each side. The proof is similar for the second claim. The formal proof, by induction, of both of these claims is left as an exercise.

**Theorem 4 [Reciprocals]**
If $a > b$ and $a, b$ have the same sign, $\frac{1}{a} < \frac{1}{b}$. Conversely, if $b$ is negative and $a$ positive, then clearly the inequality need not be reversed.

**Proof:**

Let $a > b$, then $a - b$ is positive and $b - a$ is negative. Then $\frac{b-a}{ab} = \frac{1}{a} - \frac{1}{b}$ is negative (since $ab$ is positive) and therefore $\frac{1}{a} < \frac{1}{b}$.

**Example 3**
Prove that $\frac{1}{x^2+2} < \frac{1}{x^2}$

**Solution:**

Since $x^2 + 2 > x^2$ for all $x$ and both expressions are always positive we reverse the inequality and take reciprocals.

**Theorem 5**
If $a > b > 0$, and if $p$, $q$ are positive integers then $\sqrt[q]{a} > \sqrt[q]{b}$ and therefore $a^{\frac{p}{q}} > b^{\frac{p}{q}}$.

**Proof: (by contradiction)**

Let $a > b$ and suppose that $\sqrt[q]{a} < \sqrt[q]{b}$. Raising both sides to the power of $q$ we have $a < b$, which is a contradiction. Therefore $\sqrt[q]{a} > \sqrt[q]{b}$, and by raising this to the power of $p$ we have $a^{\frac{p}{q}} > b^{\frac{p}{q}}$.

Using Theorems 4 and 5 we may now handle inequalities containing any rational exponents.

**Example 4**
Suppose that $x$ is positive and solve $\frac{1}{\sqrt{x^5}} > \frac{243}{32}$.

**Solution:**

Taking reciprocals we have $\sqrt{x^5} < \frac{243}{32}$. Now taking the fifth root we have $\sqrt{x} < \frac{3}{2}$, therefore $x < \frac{9}{4}$.

## Solving Polynomial Inequalities

In some cases polynomial inequalities can be solved by the direct application of the theorems discussed above. More generally, however, we will see that the solutions to polynomial inequations can be found by looking at the related polynomial equation.

**Example 1**
Solve $x^2 - 3x > -2$.

**Solution:**

Moving everything to one side and considering the related equation we have $x^2 - 3x + 2 = 0$. The roots are $x = 1, 2$. The function can *only* change signs at these values so it is sufficient to examine the cases $x < 1$, $1 < x < 2$, and

$x > 2$ for satisfaction of the original inequation. Testing representative values in these cases (say, $x = 0$, $x = 1.5$, $x = 3$) establishes that the solutions are $x < 1$ and $x > 2$.

If the related equation has imaginary roots we may ignore them in determining the cases to be examined. Imaginary root pairs arise as roots of quadratic factors that are never zero. Since the quadratic factor is always positive or always negative, the function does not change signs at those roots.

**Example 2**
Solve $x^3 - 2x^2 + x - 2 < 0$

**Solution**

The roots of the related equation are $x = i, -i, 2$. Therefore either $x > 2$ or $x < 2$. Using $x = 0, 3$ to test the cases we see that $x < 2$ since $x = 0$ satisfies the inequality and $x = 3$ does not.

## AM-GM Inequality

**Theorem [AM-GM].**
For $a$, $b$ positive unequal numbers we have, $\frac{a+b}{2} > \sqrt{ab}$.

**Proof:**

The square of every real quantity is positive, and therefore greater than zero. Thus $(\sqrt{a} - \sqrt{b})^2$ is positive;

$$\begin{aligned} a - 2\sqrt{a}\sqrt{b} + b &> 0; \\ a + b &> 2\sqrt{ab}; \\ \frac{a+b}{2} &> \sqrt{ab}. \end{aligned}$$

That is, the arithmetic mean of two positive quantities is greater than their geometric mean. The inequality becomes an equality when the quantities are equal.

**Theorem [AM-GM for general n].**
Let $x_1, x_2, \cdots, x_n$ be positive quantities. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdot x_2 \cdots x_n}.$$

**Proof:**

The proof is by a special induction. We have shown the case $n = 2$. We will show that if the claim holds for $n = k$ then the case $n = 2k$ holds too. Next we will show that the $n = k + 1$ case implies the $n = k$ case. Thus starting from the basis case $n = 2$ we shall imply $n = 4, 3, 6, 5, 8, 7, 10, 9 \ldots$ by doubling and back-stepping.

Suppose that the claim holds for $n = k$. Then consider;

$$\begin{aligned}\frac{x_1 + x_2 + \cdots + x_k + x_{k+1} + \cdots + x_{2k}}{2k} &\geq \frac{\sqrt[k]{x_1 \cdot x_2 \cdots x_k} + \sqrt[k]{x_{k+1} \cdot x_{k+2} \cdots x_{2k}}}{2} \\ &> \sqrt[2k]{x_1 \cdot x_2 \cdots x_k \cdot x_{k+1} \cdot x_{k+2} \cdots x_{2k}}\end{aligned}$$

Suppose now that the claim holds for $n = k + 1$ and consider;

$$\begin{aligned}a = \frac{x_1 + x_2 + \cdots + x_k}{k} &= \frac{x_1 + x_2 + \cdots + x_k + a}{k+1} \\ &\geq \sqrt[k+1]{(x_1 \cdot x_2 \cdots x_k) \cdot a}\end{aligned}$$

Now, raising both sides to the power of $k + 1$, we have

$$\begin{aligned}a^{k+1} &\geq (x_1 \cdot x_2 \cdots x_k) \cdot a \\ a^k &\geq x_1 \cdot x_2 \cdots x_k.\end{aligned}$$

So,

$$\frac{x_1 + x_2 + \cdots + x_k}{k} \geq \sqrt[k]{x_1 \cdot x_2 \cdots x_k}.$$

The theorem is thus proved for all $n$. The student should note aditionally that it is clear that equality is attained when all $x$'s are equal.

The AM-GM inequality will be found very useful, especially in the case of inequalities in which the letters are involved symmetrically.

**Example 1.**
If $a, b, c$ denote positive quantities, prove that

$$a^2 + b^2 + c^2 > bc + ca + ab;$$

$$2(a^3 + b^3 + c^3) > bc(b + c) + ca(c + a) + ab(a + b).$$

**Solution:**

Using the AM-GM inequality, we have

$$\begin{aligned}b^2 + c^2 &> 2bc; \\ c^2 + a^2 &> 2ca; \\ a^2 + b^2 &> 2ab; \\ \text{whence by addition, } a^2 + b^2 + c^2 &> bc + ca + ab.\end{aligned}$$

This result is in fact true for any real $a, b, c$. From the first line above we have

$$\begin{aligned}b^2 - bc + c^2 &> bc; \\ \text{so, } \quad b^3 + c^3 &> bc(b + c).\end{aligned}$$

By writing down similar inequalities and adding we obtain

$$2(a^3 + b^3 + c^3) > bc(b + c) + ca(c + a) + ab(a + b)$$

It should be noted that since we multiply by factors such as $(b + c)$ above we do require that they be positive in order that the inequality need not be reversed.

**Example 2.**

If $x$ may have any real value find which is the greater, $x^3 + 1$ or $x^2 + x$.

**Solution:**

$$\begin{aligned} x^3 + 1 - (x^2 + x) &= x^3 - x^2 - (x - 1) \\ &= (x^2 - 1)(x - 1) \\ &= (x - 1)^2(x + 1). \end{aligned}$$

Now $(x - 1)^2$ is positive, hence the inequality depends only on whether $(x + 1)$ is positive or negative. That is,

$$\begin{aligned} x^3 + 1 > (x^2 + x) &\quad \text{iff} \quad x > -1 \\ x^3 + 1 < (x^2 + x) &\quad \text{iff} \quad x < -1 \\ x^3 + 1 = (x^2 + x) &\quad \text{iff} \quad x = -1. \end{aligned}$$

**Theorem [Constrained Optima].**

Let $a$ and $b$ be two positive quantities, $S$ their sum and $P$ their product; then from the identity

$$4ab = (a + b)^2 - (a - b)^2,$$

we have

$$4P = S^2 - (a - b)^2, \text{ and } S^2 = 4P + (a - b)^2.$$

Hence, if $S$ is given, $P$ is greatest when $a = b$ and if $P$ is given, $S$ is least when $a = b$.

That is, *if the sum of two positive quantities is given, their product is greatest when they are equal; and if product of two positive quntities is given, their sum is least when they are equal.*

**Application.**

*To find the greatest value of a product the sum of whose factors is constant.*

Let there be $n$ factors $a, b, c, \cdots, k$, and suppose that their sum is constant and equal to $s$.

Consider the product $abc \cdots k$, and suppose that $a$ and $b$ are any two unequal factors. If we replace the two unequal factors $a$, $b$ by the two equal factors $\frac{a+b}{2}$ and $\frac{a+b}{2}$ the product is increased while the sum remains unaltered; hence so long as the product contains two unequal factors it can be increased without altering the sum of the factors; therefore the product is greatest when all the factors are equal. In this case the value of each of the $n$ factors is $\frac{s}{n}$, and the greatest value of the product is $\left(\frac{s}{n}\right)^n$, or

$$\left(\frac{a+b+c+\cdots+k}{n}\right)^n.$$

**Corollary.**
If $a, b, c, \cdots k$ are unequal,

$$\left(\frac{a+b+c+\cdots+k}{n}\right)^n > abc\cdots k;$$

that is

$$\frac{a+b+c+\cdots+k}{n} > (abc\cdots k)^{\frac{1}{n}}.$$

This result is precisely the AM-GM inequality of the previous section.

**Example 3.**
Show that $(1^r+2^r+3^r+\cdots+n^r)^n > n^n(n!)^r$; where $r$ is any real quantity.

**Solution:**

Since

$$\begin{aligned}\frac{1^r+2^r+3^r+\cdots+n^r}{n} &> (1^r2^r3^r\cdots n^r)^{\frac{1}{n}};\\ \left(\frac{1^r+2^r+3^r+\cdots+n^r}{n}\right)^n &> 1^r2^r3^r\cdots n^r;\\ &> (n!)^r.\end{aligned}$$

**Application.**
*To find the greatest value of* $a^mb^nc^p\cdots$ *when* $a+b+c+\cdots$ *is constant;* $m, n, p, \cdots$ *being positive integers.*

Since $m, n, p, \cdots$ are constants, the expression $a^mb^nc^p\cdots$ will be greatest when $\left(\frac{a}{m}\right)^m\left(\frac{b}{n}\right)^n\left(\frac{c}{p}\right)^p\cdots$ is greatest. But this last expression is the product of $m+n+p+\cdots$ factors whose sum is $m\left(\frac{a}{m}\right)+n\left(\frac{b}{n}\right)+p\left(\frac{c}{p}\right)+\cdots$, or $a+b+c+\cdots$, and therefore constant. Hence $a^mb^nc^p\cdots$ will be greatest when the factors

$$\frac{a}{m}, \frac{b}{n}, \frac{c}{p}, \cdots$$

are all equal, that is when

$$\frac{a}{m}=\frac{b}{n}=\frac{c}{p}=\frac{a+b+c+\cdots}{m+n+p+\cdots}.$$

Thus the greatest value is

$$m^mn^np^p\cdots\left(\frac{a+b+c+\cdots}{m+n+p+\cdots}\right)^{m+n+p+\cdots}$$

**Example 4.**
Find the greatest value of $(a+x)^3(a-x)^4$ for any real value of $x$ numerically less than $a$.

**Solution:**

The given expression is greatest when $\left(\frac{a+x}{3}\right)^3\left(\frac{a-x}{4}\right)^4$ is greatest; but the sum of the factors of this expression is $3\left(\frac{a+x}{3}\right)+4\left(\frac{a-x}{4}\right)=2a$; hence $(a+x)^3(a-x)^4$ is greatest when $\left(\frac{a+x}{3}\right)=\left(\frac{a-x}{4}\right)$, or $x=-\frac{a}{7}$.

Thus the greatest value is $\frac{6^3 \cdot 8^4}{7^7}a^7$.

The determination of maximum and minimum values may often be more simply effected by the solution of a quadratic equation than by the foregoing methods. Instances of this have already occurred; we add a further example.

**Example 5.**
Divide an odd integer into two integral parts whose product is a maximum.

**Solution:**

Denote the integer by $2n+1$; the two parts by $x$ and $2n+1-x$; and the product by $y$, then $(2n+1)x-x^2=y$ ,whence, by the quadratic formula,

$$2x=(2n+1)\pm\sqrt{(2n+1)^2-4y};$$

but the quantity under the radical must be positive, and therefore $y$ cannot be greater than $\frac{1}{4}(2n+1)^2$, or $n^2+n+\frac{1}{4}$; and since $y$ is integral its greatest value must be $n^2+n$; in which case $x=n+1$, or $n$; thus the two parts are $n$ and $n+1$.

**Example 6.**
Find the minimum value of $\frac{(a+x)(b+x)}{c+x}$.

**Solution:**

Put $c+x=y$; then

$$\begin{aligned}\frac{(a+x)(b+x)}{c+x} &= \frac{(a-c+y)(b-c+y)}{y};\\ &= \frac{(a-c)(b-c)}{y}+y+a-c+b-c;\\ &= \left(\frac{\sqrt{(a-c)(b-c)}}{\sqrt{y}}-\sqrt{y}\right)^2+a-c+b-c+2\sqrt{(a-c)(b-c)}.\end{aligned}$$

Hence the expression is a minimum when the square term is zero; that is when $y=\sqrt{(a-c)(b-c)}$.

Thus the minimum value is

$$a-c+b-c+2\sqrt{(a-c)(b-c)};$$

and the corresponding value of $x$ is $\sqrt{(a-c)(b-c)} - c$.

## The Values of Rational Functions

By solving certain inequalities that arise from the discriminant, we may determine the values that are attained by rational functions.

**Example 1.**
If $x$ is a real quantity, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6.

**Solution:**

Let $y = \frac{x^2+2x-11}{2(x-3)}$. This may be rewritten as $x^2 + 2x(1-y) + (6y-11) = 0$. This is a quadratic equation, and in order that $x$ may have real values the discriminant $b^2 - 4ac = 4(1-y)^2 - 4(6y-11)$ must be non-negative; or dividing by 4 and simplifying, $y^2 - 8y + 12 = (y-6)(y+2)$ must be non-negative. Hence the factors of this product must be both non-positive, or both non-negative. In the former case, $y \leq -2$, and in the latter case, $y \geq 6$. Therefore $y$ cannot lie between $-2$ and 6, but may have any other value.

In this example it will be noticed that the quadratic *expression* $y^2 - 8y + 12$ is non-negative so long as $y$ does not lie between the roots of the corresponding quadratic *equation* $y^2 - 8y + 12 = 0$. This is a particular case of the following general proposition.

**Theorem 1.**
For all real values of $x$ the expression $ax^2 + bx + c$ has the same sign as $a$, except when the roots of the equation $ax^2 + bx + c = 0$ are real and $x$ is equal to either or lies between them.

**Proof:**

CASE I.
Suppose that the roots of the equation $ax^2 + bx + c = 0$ are real and unequal. Denote them by $\alpha$ and $\beta$, and let $\alpha$ be the greater. Then

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a(x^2 - (\alpha+\beta)x + \alpha\beta) \\ &= a(x-\alpha)(x-\beta). \end{aligned}$$

Now if $x$ is greater than $\alpha$, the factors $x - \alpha$ and $x - \beta$ are both positive; and if $x$ is less than $\beta$, the factors $x - \alpha$ and $x - \beta$ are both negative. Therefore in each case the expression $(x-\alpha)(x-\beta)$ is positive, and $a^2 + bx + c$ has the same sign as $a$. But if $x$ has a value lying between $\alpha$ and $\beta$, the expression $(x-\alpha)(x-\beta)$ is negative, and the sign of $ax^2 + bx + c$ is opposite to that of $a$. Of course, if $x = \alpha$ or $x = \beta$, then $ax^2 + bx + c = 0$.

CASE II.
If the roots are real and both equal to $\alpha$, then $ax^2 + bx + c = a(x-\alpha)^2$, and $(x-\alpha)^2$ is positive for all real values of $x \neq \alpha$. Hence $ax^2+bx+c$ has the same sign as $a$ unless $x = \alpha$, in which case its value is 0.

CASE III.
Suppose that the equation $ax^2+bx+c=0$ has complex roots. Then

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= a\left(\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right). \end{aligned}$$

But $b^2-4ac$ is negative since the roots are complex. Hence $\frac{4ac-b^2}{4a^2}$ is positive, and the expression $(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a^2}$ is positive for all real values of $x$. Therefore $ax^2+bx+c$ has the same sign as $a$.

From the preceding result, it follows that the expression $ax^2+bx+c$ will always have the same sign whatever real value $x$ may have, provided that $b^2-4ac$ is negative. If this condition is satisfied, the expression is positive or negative according as $a$ is positive or negative.

Conversely, in order that the expression $ax^2+bx+c$ may be always positive, $b^2-4ac$ must be negative and $a$ must be positive; and in order that $ax^2+bx+c$ may be always negative, $b^2-4ac$ must be negative and $a$ must be negative.

**Example 2.**
Find the limits between which $a$ must lie in order that $\frac{ax^2-7x+5}{5x^2-7x+a}$ may be capable of all values, $x$ being any real quantity.

**Solution:**

Put $y=\frac{ax^2-7x+5}{5x^2-7x+a}$. Then $(a-5y)x^2-7x(1-y)+(5-ay)=0$. In order that the values of $x$ found from this quadratic may be real, the expression

$$49(1-y)^2-4(a-5y)(5-ay) = (49-20a)y^2+2(2a^2+1)y+(49-20a)$$

must be non-negative. Hence $(2a^2+1)^2-(49-20a)^2$ must be negative or zero, and $(49-20a)$ must be positive. Now

$$\begin{aligned} (2a^2+1)^2-(49-20a)^2 &= ((2a^2+1)-(49-20a))((2a^2+1)+(49-20a)) \\ &= (2a^2+20a-48)(2a^2-20a+50) \\ &= 4(a^2+10a-24)(a^2-10a+25) \\ &= 4(a+12)(a-2)(a-5)^2. \end{aligned}$$

The last expression is negative as long as $-12 < a < 2$, and for such values $49-20a$ is positive; the expression is zero when $a=-12$, 2 or 5, but $49-20a$ is negative when $a=5$. Hence $-12 \leq a \leq 2$.

# EXERCISES VII

1. Prove that $(ab + xy)(ax + by) \geq 4abxy$.
2. Prove that $(b + c)(c + a)(a + b) \geq 8abc$.
3. Show that the sum of any real positive quantity and its reciprocal is never less than 2.
4. If $a^2 + b^2 = 1$, and $x^2 + y^2 = 1$, show that $ax + by \leq 1$.
5. If $a^2 + b^2 + c^2 = 1$, and $x^2 + y^2 + z^2 = 1$, show that $ax + by + cz \leq 1$.
6. If $a \geq b$, show that $a^a b^b \geq a^b b^a$, and $\frac{b}{a} \leq \frac{1+b}{1+a}$.
7. Show that $(x^2y + y^2z + z^2x)(xy^2 + yz^2 + zx^2) \geq 9x^2y^2z^2$.
8. Find which is the greater $3ab^2$ or $a^3 + 2b^3$.
9. Prove that $a^3b + ab^3 \leq a^4 + b^4$.
10. Prove that $6abc \leq bc(b + c) + ca(c + a) + ab(a + b)$.
11. Show that $b^2c^2 + c^2a^2 + a^2b^2 \geq abc(a + b + c)$.
12. Which is the greater $x^3$ or $x^2 + x + 2$ for positive values of $x$?
13. Show that $x^3 + 13a^2x \geq 5ax^2 + 9a^3$, if $x > a$.
14. Find the greatest value of $x$ in order that $7x^2 + 11$ may be greater than $x^3 + 17x$.
15. Find the minimum value of $x^2 - 12x + 40$, and the maximum value of $24x - 8 - 9x^2$.
16. Show that $(n!)^2 > n^n$, and $2 \cdot 4 \cdot 6 \cdots 2n < (n + 1)^n$.
17. Show that $(x + y + z)^3 \geq 27xyz$.
18. Show that $n^n > 1 \cdot 3 \cdot 5 \cdots (2n - 1)$.
19. If $n$ be a positive integer greater than 2, show that $2^n > 1 + (\sqrt{2})^{n-1}$.
20. Show that $(n!)^3 < n^n \left(\frac{n+1}{2}\right)^{2n}$.
21. Show that $xyz \geq (y + z - x)(z + x - y)(x + y - z)$.
22. Show that $(x + y + z)^3 \geq 27(y + z - x)(z + x - y)(x + y - z)$.
23. Find the minimum value of $\frac{(5+x)(2+x)}{1+x}$.

24. Find the maximum value of $(7-x)^4(2+x)^5$ when $x$ lies between 7 and -2.

---

25. Determine the limits between which $n$ must lie in order that the equation

$$2ax(ax+nc)+(n^2-2)c^2=0$$

may have real roots.

26. If $x$ is real, prove that $-\frac{1}{11} \leq \frac{x}{x^2-5x+9} \leq 1$.

27. Show that $\frac{1}{3} \leq \frac{x^2-x+1}{x^2+x+1} \leq 3$ for all real values of $x$.

28. If $x$ is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

29. Find the equation whose roots are $\frac{\sqrt{a}}{\sqrt{a}\pm\sqrt{a-b}}$.

30. If $\alpha$ and $\beta$ are roots of the equation $x^2-px+q=0$, find the value of

    (a) $\alpha^2(\alpha^2\beta^{-1}-\beta)+\beta^2(\beta^2\alpha^{-1}-\alpha)$;

    (b) $(\alpha-p)^{-4}+(\beta-p)^{-4}$.

31. If the roots of $lx^2+nx+n=0$ are in the ratio of $p:q$, prove that $\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}=\sqrt{\frac{n}{l}}$.

32. Show that if $x$ is real, the expression $\frac{(x+m)^2-4mn}{2(x-n)}$ admits of all values except such as lie between $2n$ and $2m$.

33. If the roots of the equation $ax^2+2bx+c=0$ are $\alpha$ and $\beta$, and those of the equation $Ax^2+2Bx+C=0$ are $\alpha+\delta$ and $\beta+\delta$, prove that $\frac{b^2-ac}{a^2}=\frac{B^2-AC}{A^2}$.

34. Show that the expression $\frac{px^2+3x-4}{p+3x-4x^3}$ will be capable of all values when $x$ is real, provided that $1 \leq p \leq 7$.

35. Find the greatest value of $\frac{x+2}{2x^2+3x+6}$ for real values of $x$.

36. Show that if $x$ is real, the expression $\frac{x^2-bc}{2x-b-c}$ has no real values between $b$ and $c$.

37. Prove that the roots of $ax^2+2bx+c=0$ are real and unequal if and only if the roots of $(a+c)(ax^2+2bx+c)=2(ac-b^2)(x^2+1)$ are complex.

38. Show that the expression $\frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$ will be capable of all values when $x$ is real, if $a^2-b^2$ and $c^2-d^2$ have the same sign.

## Solutions to Exercises VII

2. We have $\frac{b+c}{2} \geq \sqrt{bc}$ so that $b+c \geq 2\sqrt{bc}$. Similarly, $c+a \geq 2\sqrt{ca}$ and $a+b \geq 2\sqrt{ab}$. Hence $(b+c)(c+a)(a+b) \geq 8abc$.

4. We have $a^2+x^2 \geq 2ax$ and $b^2+y^2 \geq 2by$. Thus, $a^2+b^2+x^2+y^2 \geq 2(ax+by)$. Hence, $1 \geq ax+by$.

6. We have $a^a b^b - a^b b^a = (ab)^b(a^{a-b} - b^{a-b}) > 0$. Also, $b(1+a) = b+ab < a+ab = a(1+b)$ so that $\frac{b}{a} < \frac{1+b}{1+a}$.

8. We have $a^3+2b^3 = a^3+b^3+b^3 \geq 3(\sqrt[3]{a^3b^3b^3}) = 3ab^2$.

10. We have

$$\begin{aligned} bc(b+c)+ca(c+a)+ab(a+b) &= c(b^2+a^2)+b(c^2+a^2)+a(b^2+c^2) \\ &\geq 2c\sqrt{a^2b^2}+2b\sqrt{c^2a^2}+2a\sqrt{b^2c^2} \\ &= 6abc. \end{aligned}$$

12. We have $x^3-x^2-x-2 = (x-2)(x^2+x+1)$. For $x > 0$, $x^2+x+1 > 0$. Hence $x^3 > x^2+x+2$ if and only if $x > 2$.

14. We have

$$\begin{aligned} x^3-7x^2+17x-11 &= (x-1)(x^2-8x+11) \\ &= (x-1)((x-4)^2-5) \\ &= (x-1)(x-4+\sqrt{5})(x-4-\sqrt{5}). \end{aligned}$$

For $x > 1$, all three factors are positive. It follows that the maximum value of $x$ for which $7x^2+11 \geq x^3+17x$ is $x = 1$.

16. We have $n = 1(n) \leq 2(n-1) \leq 3(n-2) \leq \cdots$. Hence $n^2 < (n!)^2$ since not all equalities can hold simultaneously. On the other hand, $2(2n) < 4(2n-2) < 6(2n-4) < \cdots < (n+1)^2$. Hence $2 \times 4 \times 6 \times \cdots \times 2n < (n+1)^n$.

18. We have $1(2n-1) < 3(2n-3) < 5(2n-5) < \cdots < n^2$. Hence $1 \times 3 \times \cdots \times (2n-1) < n^n$.

20. We have $1(n) < 2(n-1) < 3(n-2) < \cdots < (\frac{n+1}{2})^2$. Hence $(n!)^2 < (\frac{n+1}{2})^{2n}$. Since $n! < n^n$, we have $(n!)^3 < n^n(\frac{n+1}{2})^{2n}$.

22. We have $(y+z-x)(z+x-y)(x+y-z) \leq \left(\frac{x+y+z}{3}\right)^3$ by the AM-GM inequality. Thus $(x+y+z)^3 \geq 27(y+z-x)(z+x-y)(x+y-z)$.

24. The expression $(7-x)^4(2+x)^5$ takes on its maximum value at the same time as the expression $(35-5x)^4(8+4x)^5$. In the latter expression, the sum of the nine factors is 180. It follows that $35-5x = 20$ so that $x = 3$. Hence the maximum value of the original expression is $4^4 5^5$.

26. Let $y = \frac{x}{x^2-5x+9}$. Then $yx^2 - (5y+1)x + 9y = 0$. In order for $x$ to be real, we must have $(5y+1)^2 - 4y(9y) \geq 0$. This is equivalent to $1 + 10y - 11y^2 \geq 0$ or $(1-y)(1+11y) \geq 0$. If $y > 1$, the first factor is negative but the second is positive. If $y < -\frac{1}{11}$, the first factor is positive but the second is negative. Hence we must have $-\frac{1}{11} \leq y \leq 1$.

28. Let $y = \frac{x^2+34x-71}{x^2+2x-7}$. Then $(y-1)x^2 + 2(y-17)x - (7y-71) = 0$ In order for $x$ to be real, we must have $4(y-17)^2 - 4(y-1)(7y-71) \geq 0$. This is equivalent to $8y^2 - 112y + 360 \geq 0$ or $8(y-5)(y-9) \geq 0$. If $5 < y < 9$, then $y - 5 > 0$ but $y - 9 < 0$. Hence $y$ has no value between 5 and 9.

30. We have $\alpha+\beta = p$ and $\alpha\beta = q$. Hence $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = p^2 - q$,

$$\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta) = p^3 - 3pq,$$

$$\begin{aligned}\alpha^4 + \beta^4 &= (\alpha+\beta)^4 - 4\alpha\beta(\alpha^2+\beta^2) - 6(\alpha\beta)^2 \\ &= p^4 - 4q(p^2 - q) - 6q^2 \\ &= p^4 - 4p^2q + 2q^2\end{aligned}$$

and

$$\begin{aligned}\alpha^5 + \beta^5 &= (\alpha+\beta)^5 - 5\alpha\beta(\alpha^3+\beta^3) - 10(\alpha\beta)^2(\alpha+\beta) \\ &= p^5 - 5q(p^3 - 3pq) - 10pq^2 = p^5 - 5p^2q + 5pq^2.\end{aligned}$$

(a) We have

$$\begin{aligned}\alpha^2\left(\frac{\alpha^2}{\beta} - \beta\right) + \beta^2\left(\frac{\beta^2}{\alpha} - \alpha\right) &= \frac{\alpha^5 + \beta^5 - (\alpha\beta)^2(\alpha+\beta)}{\alpha\beta} \\ &= \frac{p^5 - 5p^2q + 5pq^2 - pq^2}{q} \\ &= \frac{p(p^2-q)(p^2-4q)}{q}.\end{aligned}$$

(b) We have

$$\begin{aligned}&\frac{1}{(\alpha-p)^4} + \frac{1}{(\beta-p)^4} \\ = \; &\frac{\alpha^4 + \beta^4 - 4p(\alpha^3+\beta^3) - 6p^2(\alpha^2+\beta^2) - 4p^3(\alpha+\beta) + 2p^2}{(\alpha\beta - (\alpha+\beta)p + p^2)^4} \\ = \; &\frac{p^4 - 4p^2q + 2q^2 - 4p(p^3 - 3pq) + 6p^2(p^2 - 2q) - 4p^4 + 2p^4}{(q - p^2 + p^2)^4} \\ = \; &\frac{p^4 - 4p^2q + 2q^2}{q^4}.\end{aligned}$$

32. Let $y = \frac{(x+m)^2-4mn}{2(x-n)}$. Then $x^2+2(m-y)+m^2-4mn-2ny=0$. In order for $x$ to be real, we must have $4(m-y)^2-4(m^2-4mn+2ny) \geq 0$. This is equivalent to $y^2-2(m+n)y+4mn \geq 0$ or $(y-2m)(y-2n) \geq 0$. If $y$ lies between $2m$ and $2n$, one factor is positive but the other is negative. This is a contradiction. For any other value of $y$, there are two values of $x$, the roots of the quadratic equation in $x$ given earlier, which would yield that particular value of $y$.

34. Let $y = \frac{px^2+3x-4}{p+3x-4x^2}$. Then $(p+4y)x^2+3(1-y)x-(4+py)=0$. In order for $x$ to be real, we must have $9(1-y)^2+4(p+4y)(4+py) \geq 0$. This is equivalent to

$$(16p+9)y^2+2(2p^2+23)y+(16p+9) \geq 0.$$

Now

$$\begin{aligned}
& (16p+9)((16p+9)y^2+2(2p^2+23)y+(16p+9)) \\
= \; & (16p+9)^2y^2+2(2p^2+23)(16p+9)y+(2p^2+23)^2+(16p+9)^2 \\
& -(2p^2+23)^2 \\
= \; & ((16p+9)y+(2p^2+23))^2+(2p^2+16p+32)(-2p^2+16p-14) \\
= \; & ((16p+9)y+(2p^2+23))^2+4(p+4)^2(p-1)(7-p).
\end{aligned}$$

If $1 \leq p \leq 7$, then $16p+9>0$ and $(p-1)(7-p) \geq 0$. The desired inequality follows.

36. Let $y = \frac{x^2-bc}{2x-b-c}$. Then $x^2-2yx+by+cy-bc=0$. In order for $x$ to be real, we must have $4y^2-4(by+cy-bc) \geq 0$. This is equivalent to $(y-b)(y-c) \geq 0$. Hence $y$ cannot lie between $b$ and $c$.

38. Let $y = \frac{(ax-b)(dx-c)}{(bx-a)(cx-d)}$. Then $(ad-bcy)x^2-(ac+bd)(1-y)x+(bc-ady)=0$. In order for $x$ to be real, we must have $(ac+bd)^2(1-y)^2-4(ad-bcy)(bc-ady) \geq 0$. This is equivalent to $(ac-bd)^2y^2-2(a^2d^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2)y+(ac-bd)^2 \geq 0$. We may rewrite this expression as $((ac-bd)y-\frac{a^2c^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2}{ac-bd})^2+(ac-bd)^2-(\frac{a^2c^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2}{ac-bd})^2$. This is non-negative if so is $(a^2c^2-2abcd+b^2d^2)^2-(a^2c^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2)^2$. This is a difference of two squares. Therefore, it is a product of two factors. One of them is $a^2c^2-2abcd+b^2d^2-a^2c^2-2abcd-b^2d^2+2a^2d^2+2b^2c^2=2(ad-bc)^2 \geq 0$. The other factor is $a^2d^2-2abcd+b^2d^2+a^2c^2+2abcd+b^2d^2-2a^2d^2-2b^2c^2=2(a^2-b^2)(c^2-d^2)$, which is positive if $a^2-b^2$ and $c^2-d^2$ have the same sign.

## Answers to Exercises VII

15. 4; 8.

23. 9, when $x=1$.

25. $-2 \leq n \leq 2$.

29. $bx^2-2ax+a=0$.

35. $\frac{1}{3}$.

# CHAPTER VIII — SYMMETRIC IDENTITIES

A function is said to be *symmetrical* with respect to its variables when its value is unaltered by the interchange of any pair of them; thus $x+y+z$, $bc+ca+ab$, and $x^3+y^3+z^3-3xyz$ are symmetrical functions of the first, second, and third degrees respectively.

It is worthy of note that the only symmetrical function of the first degree in $x$, $y$, and $z$ is of the form $M(x+y+z)$ where $M$ is independent of $x$, $y$, and $z$.

It easily follows from the definition that the sum, difference, product and quotient of any two symmetrical expressions must also be symmetrical expressions. The recognition of this principle is of great use in checking the accuracy of algebraical work, and in some cases enables us to dispense with much of the labour of calculation.

**Example 1.**
Expand $(x+y+z)^3$.

**Solution:**

We know that the expansion must be a homogeneous function of three dimensions, and therefore of the form $x^3+y^3+z^3+A(x^2y+xy^2+y^2z+yz^2+z^2x+zx^2)+Bxyz$, where $A$ and $B$ are quantities independent of $x$, $y$, and $z$. Put $z=0$. Since the coefficient of $x^2y$ in $(x+y)^3$ is 3, we must have $A=3$. Put $x=y=z=1$, and we get $27=3+3(6)+B$; whence $B=6$. Thus $(x+y+z)^3=x^3+y^3+z^3+3x^2y+3xy^2+3y^2z+3yz^2+3z^2x+3zx^2+6xyz$.

**Example 2.**
Prove that $(a+b)^5-a^5-b^5=5ab(a+b)(a^2+ab+b^2)$.

**Solution:**

Denote the expression on the left by $E$; then $E$ is a function of $a$ which vanishes when $a=0$; hence $a$ is a factor of $E$ ; similarly $b$ is a factor of $E$. Again $E$ vanishes when $a=-b$, that is $a+b$ is a factor of $E$; and therefore $E$ contains $ab(a+b)$ as a factor. The remaining factor must be of two dimensions, and, since it is symmetrical with respect to $a$ and $b$, it must be of the form $Aa^2+Bab+Ab^2$; thus

$$(a+b)^5-a^5-b^5=ab(a+b)(Aa^2+Bab+Ab^2),$$

where $A$ and $B$ are independent of $a$ and $b$. Putting $a=b=1$, we have $15=2A+B$; putting $a=2$ and $b=-1$, we have $15=5A-2B$; whence $A=5$ and $B=5$; thus the required result at once follows.

**Example 3.**
Show that $(x+y+z)^5-x^5-y^5-z^5=5(y+z)(z+x)(z+y)(x^2+y^2+z^2+yz+zx+xy)$.

**Solution:**

Denote the expression on the left by $E$; then $E$ vanishes when $y = -z$, and therefore $y + z$ is a factor of $E$; similarly $z + x$ and $x + y$ are factors; therefore $E$ contains $(y+z)(z+x)(z+y)$ as a factor. Also, since $E$ is of the fifth degree the remaining factor is of the second degree, and, since it is symmetrical in $x, y$ and $z$, it must be of the form

$$A(x^2 + y^2 + z^2) + B(yz + zx + xy).$$

Put $x = y = z = 1$; thus $10 = A + B$; put $x = 2, y = 1$ and $z = 0$; thus $35 = 5A + 2B$; whence $A = B = 5$, and we have the required result.

A function is said to be *alternating* with respect to its variables when its sign but not its absolute value is altered by the interchange of any pair of them. Thus $x - y$ and $a^2(b-c) + b^2(c-a) + c^2(a-b)$ are alternating functions.

It is evident that there can be no linear alternating function involving more than two variables, and also that the product of a symmetrical function and an alternating function is an alternating function.

**Example 4.**
Factorize $(b^3 + c^3)(b - c) + (c^3 + a^3)(c - a) + (a^3 + b^3)(a - b)$.

**Solution:**

Denote the expression by $E$; then $E$ is a function of $a$ which vanishes when $a = b$, and therefore contains $a - b$ as a factor. Similarly, it contains the factors $b - c$ and $c - a$; thus $E$ contains $(b - c)(c - a)(a - b)$ as a factor. Also, since $E$ is of the fourth degree, the remaining factor must be of the first degree; and since it is a symmetrical function of $a$, $b$ and $c$, it must be of the form $M(a + b + c)$. Thus $E = M(b - c)(c - a)(a - b)(a + b + c)$. To obtain $M$, we may give to $a, b$ and $c$ any values that we find most convenient; thus by putting $a = 0$, $b = 1$ and $c = 2$, we find $M = 1$, and we have the required result.

**Notation [$\sum$].**

Symmetrical and alternating functions may be concisely denoted by writing down one of the terms and prefixing the symbol $\sum$; thus $\sum a$ stands for the sum of all the terms of which $a$ is the type, $\sum ab$ stands for the sum of all the terms of which $ab$ is the type; and so on. For instance, if the function involves four letters $a, b, c, d$,

$$\begin{aligned} \sum a &= a + b + c + d; \\ \sum ab &= ab + ac + ad + bc + bd + cd. \end{aligned}$$

Similarly if the function involves three letters $a, b, c$,

$$\begin{aligned} \sum a^2(b-c) &= a^2(b-c) + b^2(c-a) + c^2(a-b); \\ \sum a^2bc &= a^2bc + b^2ca + c^2ab. \end{aligned}$$

It should be noticed that when there are three letters involved $\sum a^2b$ does not consist of three terms, but of six: thus

$$\sum a^2b = a^2b + a^2c + b^2c + b^2a + c^2a + c^2b.$$

The symbol $\sum$ may also be used to imply summation with regard to two or more sets of letters; thus

$$\sum yz(b-c) = yz(b-c) + zx(c-a) + xy(a-b).$$

The main reason for introducing this notation is to enable us to express in an abridged form the products and powers of symmetrical expressions; thus

$$\begin{aligned}
(a+b+c)^3 &= \sum a^3 + 3\sum a^2b + 6abc;\\
(a+b+c+d)^3 &= \sum a^3 + 3\sum a^2b + 6\sum abc;\\
(a+b+c)^4 &= \sum a^4 + 4\sum a^3b + 6\sum a^2b^2 + 12\sum a^2bc;\\
\sum a \times \sum a^2 &= \sum a^3 + \sum a^2b.
\end{aligned}$$

### Common Forms.

We collect here for reference a list of identities in three letters which are useful in the transformation of algebraical expressions; the student should verify these identities.

$$\begin{aligned}
\sum bc(b-c) &= -(b-c)(c-a)(a-b).\\
\sum a^2(b-c) &= -(b-c)(c-a)(a-b).\\
\sum a(b^2-c^2) &= (b-c)(c-a)(a-b).\\
\sum a^3(b-c) &= -(b-c)(c-a)(a-b)(a+b+c).\\
a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-bc-ca-ab).\\
&= \frac{1}{2}(a+b+c)((b-c)^2+(c-a)^2+(a-b)^2).\\
(b-c)^3+(c-a)^3+(a-b)^3 &= 3(b-c)(c-a)(a-b).\\
(a+b+c)^3-a^3-b^3-c^3 &= 3(b+c)(c+a)(a+b).\\
\sum bc(b+c)+2abc &= (b+c)(c+a)(a+b).\\
\sum a^2(b+c)+2abc &= (b+c)(c+a)(a+b).\\
(a+b+c)(bc+ca+ab)-abc &= (b+c)(c+a)(a+b).\\
2(b^2c^2+c^2a^2+a^2b^2)-(a^4+b^4+c^4) &= (a+b+c)(b+c-a)(c+a-b)(a+b-c).
\end{aligned}$$

### Miscellaneous Identities.

Many identities can be readily established by making use of the properties of the cube roots of unity; as usual these will be denoted by 1, $\omega$ and $\omega^2$.

**Example 5.**
Show that $(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2+xy+y^2)^2$.

**Solution:**

The expression, $E$, on the left vanishes when $x=0, y=0, x+y=0$; hence it must contain $xy(x+y)$ as a factor. Putting $x = \omega y$, we have

$$\begin{aligned} E &= (1+\omega)^7 - \omega^7 - 1y^7 = (-\omega^2)^7 - \omega^7 - 1y^7 \\ &= (-\omega^2 - \omega - 1)y^7 = 0. \end{aligned}$$

Hence $E$ contains $x - \omega y$ as a factor; and similarly we may show that it contains $x - \omega^2 y$ as a factor; that is, $E$ is divisible by $(x-\omega y)(x-\omega^2 y)$ or $x^2+xy+y^2$. Further, $E$ being of seven, and $xy(x+y)(x^2+xy+y^2)$ of five dimensions, the remaining factor must be of the form $A(x^2+y^2)+Bxy$; thus

$$(x+y)^7 - x^7 - y^7 = xy(x+y)(x^2+xy+y^2)(Ax^2+Bxy+Ay^2).$$

Putting $x=y=1$, we have $21 = 2A+B$; and putting $x=2$ and $y=-1$, we have $21 = 5A-2B$; whence $A=7, B=7$; and

$$(x+y)^7 - x^7 - y^7 = 7xy(x+y)(x^2+xy+y^2)^2.$$

We have the following two factorizations

$$\begin{aligned} a^3+b^3+c^3-3abc &= (a+b+c)(a^2+b^2+c^2-bc-ca-ab) \\ a^2+b^2+c^2-bc-ca-ab &= (a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c). \end{aligned}$$

Hence $a^3+b^3+c^3-3abc$ can be resolved into three linear factors; thus

$$a^3+b^3+c^3-3abc = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c).$$

**Example 6.**
Show that the product of $a^3+b^3+c^3-3abc$ and $x^3+y^3+z^3-3xyz$ can be put into the form $A^3+B^3+C^3-3ABC$.

**Solution:**

The product is $P = (a+b+c)(a+\omega b+\omega^2 c)(a+\omega^2 b+\omega c)\times(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$. By taking these six factors in the pairs $(a+b+c)(x+y+z)$, $(a+\omega b+\omega^2 c)(x+\omega^2 y+\omega z)$ and $(a+\omega^2 b+\omega c)(x+\omega y+\omega^2 x)$, we obtain the three partial products $A+B+C$, $A+\omega B+\omega^2 C$ and $A+\omega^2 B+\omega C$, where $A = ax+by+cz$, $B = bx+cy+az$ and $C = cx+ay+bz$. Thus,

$$\begin{aligned} P &= (A+B+C)(A+\omega B+\omega^2 C)(A+\omega^2 B+\omega C) \\ &= A^3+B^3+C^3-3ABC. \end{aligned}$$

**Example 7.**
If $\alpha+\beta+\gamma=0$, show that

$$6(\alpha^5+\beta^5+\gamma^5) = 5(\alpha^3+\beta^3+\gamma^3)(\alpha^2+\beta^2+\gamma^2).$$

**Solution:**

We might employ the substitutions $\alpha = h + k, \beta = \omega h + \omega^2 k$ and $\gamma = \omega^2 h + \omega k$. Then $\alpha\beta + \beta\gamma + \gamma\alpha = -3hk$ and $\alpha\beta\gamma = -(h^3 + k^3$. We have

$$\begin{aligned}
\sum \alpha^2 &= \left(\sum \alpha\right)^2 - 2\left(\sum \beta\gamma\right) \\
&= 6hk; \\
\sum \alpha^3 &= \left(\sum \alpha\right)^3 - \left(\sum \alpha\right)\left(\sum \beta\gamma\right) + 3\alpha\beta\gamma \\
&= 3(h^3 + k^3); \\
\sum \alpha^3\beta^2 &= \left(\sum \alpha\right)\left(\sum \beta^2\gamma^2\right) - \alpha\beta\gamma\left(\sum \beta\gamma\right) \\
&= 3hk(h^3 + k^3); \\
\sum \alpha^5 &= \left(\sum \alpha^2\right)\left(\sum \alpha^3\right) - \sum \alpha^3\beta^2 \\
&= 15hk(h^3 + k^3).
\end{aligned}$$

The desired identity follows from $6(15hk(h^3 + k^3)) = 5(6hk)(3(h^3 + k^3))$.

The Notation of this chapter may be used with Vieta's Theorem as in the following example.

**Example 8.**
If $\alpha, \beta, \gamma$ and $\delta$ are the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, find the value of $\sum \alpha^2\beta$.

**Solution:**

We have $\alpha + \beta + \gamma + \delta = -p, \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta - \gamma\delta = q$ and $\alpha\beta\gamma + \alpha\beta\delta + \alpha\beta\gamma\delta + \beta\gamma\delta = -r$. From these equations we have $-pq = \sum \alpha^2\beta + 3\sum \alpha\beta\gamma = \sum \alpha^2\beta - 3r$. Hence $\sum \alpha^2\beta = 3r - pq$.

## EXERCISES VIII

Factorize

1. $a^4(b^2 - c^2) + b^4(c^2 - a^2) + c^4(a^2 - b^2)$.
2. $(a + b + c)^3 - (b + c - a)^3 - (c + a - b)^3 - (a + b - c)^3$.
3. $a(b - c)^2 + b(c - a)^2 + c(a - b)^2 + 8abc$.
4. $a(b^4 - c^4) + b(c^4 - a^4) + c(a^4 - b^4)$.
5. $(bc + ca + ab)^3 - b^3c^3 - c^3a^3 - a^3b^3$.
6. $(a + b + c)^4 - (b + c)^4 - (c + a)^4 - (a + b)^4 + a^4 + b^4 + c^4$.
7. $(a + b + c)^6 - (b + c - a)^6 - (c + a - b)^6 - (a + b - c)^6$.

8. $(x-a)^3(b-c)^3+(x-b)^3(c-a)^3+(x-c)^3(a-b)^3$.

9. $(b+c-2a)^3+(c+a-2b)^3+(a+b-2c)^2$.

10. $a(b-c)^3+b(c-a)^3+c(a-b)^3$.

11. $x^4(y+z)^2+y^4(z+x)+z^4(x+y)+2(yz+zx+xy)^3-2x^2y^2z^2$.

12. $a^2(b+c)+b^2(c+a)+c^2(a+b)-(a^3+b^3+c^3)-2abc$.

Simplify

13. $\frac{2a}{a+b}+\frac{2b}{b+c}+\frac{2c}{c+a}+\frac{(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)}$.

14. $\frac{a^2-b^2-c^2}{(a-b)(a-c)}+\frac{b^2-c^2-a^2}{(b-c)(b-a)}+\frac{c^2-a^2-b^2}{(c-a)(c-b)}$.

15. $\sum\frac{bcd}{(a-b)(a-c)(a-d)}$.

16. $\frac{a(b-c)^2}{(c-a)(a-b)}+\frac{b(c-a)^2}{(a-b)(b-c)}+\frac{c(a-b)^2}{(b-c)(c-a)}$.

17. $\frac{a^3(b+c)}{(a-b)(a-c)}+\frac{b^3(c+a)}{(b-c)(b-a)}+\frac{c^3(a+b)}{(c-a)(c-b)}$.

18. $\frac{a}{(a-b)(a-c)(x-a)}+\frac{b}{(b-c)(b-a)(x-b)}+\frac{c}{(c-a)(c-b)(x-c)}$.

19. $\frac{(a+p)(a+q)}{(a-b)(a-c)(a+x)}+\frac{(b+p)(b+q)}{(b-c)(b-a)(b+x)}+\frac{(c+p)(c+q)}{(c-a)(c-b)(c+x)}$.

20. $\sum\frac{a^4}{(a-b)(a-c)(a-d)}$.

21. Prove that $4\sum(b-c)(b+c-2a)^2=9\sum(b-c)(b+c-a)^2$.

22. Prove that $\sum(ab-c^2)(ac-b^2)=(\sum bc)(\sum bc-\sum a^2)$.

23. Prove that $abc(\sum a)^3-(\sum bc)^3=abc\sum a^3-\sum b^3c^3=(a^2-bc)(b^2-ca)(c^2-ab)$.

24. (a) Prove that $\sum(b-c)^3(b+c-2a)=0$.
    (b) Deduce that $\sum(b-c)(b+c-2a)^3=0$.

25. Prove that $(b+c)^3+(c+a)^3+(a+b)^3-3(b+c)(c+a)(a+b)=2(a^3+b^3+c^3-3abc)$.

26. If $x=b+c-a$, $y=c+a-b$ and $z=a+b-c$, show that $x^3+y^3+z^3-3xyz=4(a^3+b^3+c^3-3abc)$.

27. Prove that the value of $a^3+b^3+c^3-3abc$ is unaltered if we substitute $s-a$, $s-b$ and $s-c$ for $a$, $b$ and $c$ respectively, where $3s=2(a+b+c)$.

28. If $x + y + z = s$ and $xyz = p^2$, show that

$$\left(\frac{p}{ys} - \frac{y}{p}\right)\left(\frac{p}{zs} - \frac{z}{p}\right) + \left(\frac{p}{zs} - \frac{z}{p}\right)\left(\frac{p}{xs} - \frac{x}{p}\right) + \left(\frac{p}{xs} - \frac{x}{p}\right)\left(\frac{p}{ys} - \frac{y}{p}\right) = \frac{4}{s}.$$

29. If $(a+b+c)^3 = a^3+b^3+c^3$, show that $(a+b+c)^{2n+1} = a^{2n+1}+b^{2n+1}+c^{2n+1}$ when $n$ is a positive integer.

30. Show that $(x + y)^n - x^n - y^n$ is divisible by $xy(x^2 + xy + y^2)$, if $n$ is an odd positive integer not a multiple of 3.

31. Show that $(a + \omega b + \omega^2 c)^3 + (a + \omega^2 b + \omega c)^3 = (2a - b - c)(2b - c - a)(2c - a - b)$.

32. Find the value of $(b - c)(c - a)(a - b) + (b - \omega c)(c - \omega a)(a - \omega b) + (b - \omega^2 c)(c - \omega^2 a)(a - \omega^2 b)$.

33. Show that $(a^2 + b^2 + c^2 - bc - ca - ab)(x^2 + y^2 + z^2 - yz - zx - xy)$ may be put into the form $A^2 + B^2 + C^2 - BC - CA - AB$.

34. Show that $(a^2 + ab + b^2)(x^2 + xy + y^2)$ can be put into the form $A^2 + AB + B^2$, and find the value of $A$ and $B$.

Show that

35. $\sum(a^2 + 2bc)^3 - 3(a^2 + 2bc)(b^2 + 2ca)(c^2 + 2ab) = (a^3 + b^3 + c^3 - 3abc)^2$.

36. $\sum(a^2 - bc)^3 - 3(a^2 - bc)(b^2 - ca)(c^2 - ab) = (a^3 + b^3 + c^3 - 3abc)^2$.

37. $a^3(bz-cy)^3+b^3(cx-az)^3+c^3(ay-bx)^3 = 3abc(bz-cy)(cx-az)(ay-bx)$.

38. $(a^2 + b^2 + c^2)^3 + 2(bc + ca + ab)^3 - 3(a^2 + b^2 + c^2)(bc + ca + ab)^2 = (a^3 + b^3 + c^3 - 3abc)^2$.

39. $25((y - z)^7 + (z - x)^7 + (x - y)^7)((y - z)^3 + (z - x)^3 + (x - y)^3) = 21((y - z)^5 + (z - x)^5 + (x - y)^5)^2$.

40. $((y - z)^2 + (z - x)^2 + (x - y)^2)^3 - 54(y - z)^2(z - x)^2(x - y)^2 = 2(y + z - 2x)^2(z + x - 2y)^2(x + y - 2z)^2$.

41. $(b - c)^6 + (c - a)^6 + (a - b)^6 - 3(b - c)^2(c - a)^2(a - b)^2 = 2(a^2 + b^2 + c^2 - bc - ca - ab)^3$.

42. $(b-c)^7+(c-a)^7+(a-b)^7 = 7(b-c)(c-a)(a-b)(a^2+b^2+c^2-bc-ca-ab)^2$.

If $a + b + c = 0$, prove that

43. $2(a^4 + b^4 + c^4) = (a^2 + b^2 + c^2)^2$.

44. $a^5 + b^5 + c^5 = -5abc(bc + ca + ab)$.

45. $a^6 + b^6 + c^6 = 3a^2b^2c^2 - 2(bc + ca + ab)^3$.

46. $3(a^2 + b^2 + c^2)(a^5 + b^5 + c^5) = 5(a^3 + b^3 + c^3)(a^4 + b^4 + c^4)$.

47. $\frac{a^7+b^7+c^7}{7} = \frac{a^5+b^5+c^5}{5} \cdot \frac{a^2+b^2+c^2}{2}$.

48. $(\frac{b-c}{a} + \frac{c-a}{b} + \frac{a-b}{c})(\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}) = 9$.

If $a + b + c + d = 0$, show that

49. $\frac{a^5+b^5+c^5+d^5}{5} = \frac{a^3+b^3+c^3+d^3}{3} \cdot \frac{a^2+b^2+c^2+d^2}{2}$.

50. $(a^3+b^3+c^3+d^3)^2 = 9(bcd+cda+dab+abc)^2 = 9(bc-ad)(ca-bd)(ab-cd)$.

51. Prove that $(b^2c + c^2a + a^2b - 3abc)(bc^2 + ca^2 + ab^2 - 3abc) = (bc + ca + ab)^3 + 27a^2b^2c^2$ if $a + b + c = 0$.

52. If $a + b + c = 0$ and $x + y + z = 0$, show that $54abcxyz$ is equal to $4(ax + by + cz)^3 - 3(ax + by + cz)(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) - 2(b - c)(c - a)(a - b)(y - z)(z - x)(x - y)$.

53. Prove that $\sum(s - b)(s - c)(S^2 - a^2) + 5abcs = (s^2 - S^2)(4s^2 + S^2)$ if $2s = a + b + c$ and $2S^2 = a^2 + b^2 + c^2$.

54. Show that $(x^3+6x^2y+3xy^2-y^3)^3+(y^3+6xy^2+3x^2y-x^3)^3 = 27xy(x+y)(x^2 + xy - y^2)^3$.

55. Show that $\sum \frac{a^5}{(a-b)(a-c)(a-d)} = a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd$.

56. Factorize $2a^2b^2c^2 + (a^3 + b^3 + c^3)abc + b^3c^3 + c^3a^3 + a^3b^3$.

57. State Vieta's Theorem for $n^{th}$ degree polynomial equations using the notation of this chapter.

If $\alpha$, $\beta$ and $\gamma$ are the roots of the equation $x^3 - px^2 + qx - r = 0$, find the value of:

58. $\sum \alpha^2\beta^2$. 59. $\sum \alpha + \beta$. 60. $\sum(\frac{\alpha}{\beta} + \frac{\beta}{\alpha})$. 61. $\sum \alpha^2\beta$.

If $\alpha$, $\beta$, $\gamma$ and $\delta$ are the roots of $x^4 + px^3 + qx^2 + rx + s = 0$, find the value of:

62. $\sum \alpha^2\beta\gamma$. 63. $\sum \alpha^4$.

## Solutions to Exercises VIII

2. Denote the expression by $E$. Then $E$ is a function of $a$ which vanishes when $a = 0$. Therefore, it contains $a$ as a factor. Similarly, it contains the factors $b$ and $c$. Since $E$ is of the third degree, we have $E = Mabc$. Putting $a = b = c = 1$, we have $M = 24$. Hence $E = 24abc$.

4. Denote the expression by $E$. Then $E$ is a function of $a$ which vanishes when $a = b$. Therefore, it contains $a - b$ as a factor. Similarly, it contains the factors $b - c$ and $c - a$. Since $E$ is of the fifth degree, the remaining factor must be of the second degree; and since it is a symmetrical functions of $a$, $b$ and $c$, it must be of the form $A(a^2 + b^2 + c^2) + B(bc + ca + ab)$. Thus $E = (b - c)(c - a)(a - b)(A(a^2 + b^2 + c^2) + B(bc + ca + ab)$. To obtain $A$ and $B$, put $a = 0$, $b = 1$ and $c = 2$. We find $5A + 2B = 7$. Put $a + 0$, $b + 1$ and $c = 3$. We find $10A + 3B = 13$. Hence $A = B = 1$ and we have $E = (b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$.

6. Denote the expression by $E$. Then $E$ is a function of $a$ which vanishes when $a = 0$. Therefore, it contains $a$ as a factor. Similarly, it contains the factors $b$ and $c$. Since $E$ is of the fourth degree, the remaining factor must be of the first degree; and since it is a symmetrical functions of $a$, $b$ and $c$, it must be of the form $M(a + b + c)$. Thus $E = M(a + b + c)abc$. To obtain $M$, put $a = b = c = 1$. We find $M = 12$, and we have $E = 12(a + b + c)abc$.

8. Denote the expression by $E$. Then $E$ is a function of $x$ which vanishes when $x = a$. Therefore, it contains $x - a$ as a factor. Similarly, it contains the factors $x - b$ and $x - c$. Now $E$ is also a function of $a$ which vanishes when $a = b$. Therefore, it contains $a - b$ as a factor. Similarly, it contains the factors $b - c$ and $c - a$. Since $E$ is of the sixth degree, $E = M(x-a)(x-b)(x-c)(b-c)(c-a)(a-b)$. To obtain $M$, put $a = 0$, $b = 1$ and $c = 2$. We find $2Mx(x - 1)(x - 2) = -x^3 + 8(x - 1)^3 - (x - 2)^3$. Comparing the coefficients of $x^3$ on both sides, we have $M = 3$, so that $E = 3(x - a)(x - b)(x - c)(b - c)(c - a)(a - b)$.

10. Denote the expression by $E$. Then $E$ is a function of $a$ which vanishes when $a = b$. Therefore, it contains $a - b$ as a factor. Similarly, it contains the factors $b - c$ and $c - a$. Since $E$ is of the fourth degree, the remaining factor must be of the first degree; and since it is a symmetrical functions of $a$, $b$ and $c$, it must be of the form $M(a + b + c)$. Thus $E = M(b - c)(c - a)(a - b)(a + b + c)$. To obtain $M$, put $a = 0$, $b = 1$ and $c = 2$. We find $M = 1$, and we have $E = (b-c)(c-a)(a-b)(a+b+c)$.

12. Set $a = b + c$. Then $\sum a^2(b + c) = a^3 + b^3 + 2b^2c + c^3 + 2bc^2$, $\sum a^3 = a^3 + b^3 + c^3$ and $2abc = 2b^2c + 2bc^2$. Hence $\sum a^2(b+c) - \sum a^3 - 2abc = 0$. It follows that the given expression has $b + c - a$ as a factor. Similarly,

it has the factors $c+a-b$ and $a+b-c$, so that it has the form $M(b+c-a)(c+a-b)(a+b-c)$. Putting $a=b=c=1$, we find $M=1$ so that $(b+c-a)(c+a-b)(a+b-c)$ is the desired factorization.

14. Let $E$ denote the expression. Taking common denominators, we have

$$\begin{aligned} E &= \frac{(c-b)(a^2-b^2-c^2)+(a-c)(b^2-c^2-a^2)+(b-a)(c^2-a^2-b^2)}{(a-b)(b-c)(c-a)} \\ &= \frac{2(ca^2+ab^2+bc^2)-2(a^2b+b^2c+c^2a)}{(a-b)(b-c)(c-a)}. \end{aligned}$$

The numerator vanishes when we put $a=b$. Hence it has $a-b$ as a factor. Similarly, it has the factors $b-c$ and $c-a$. Since it is of the third degree, $E$ is equal to a constant $M$. Putting $a=0$, $b=1$ and $c=2$, we find $M=2$.

16. Let $E$ denote the expression in Problem 10. Taking common denominators, our sum is equal to $\frac{E}{(b-c)(c-a)(a-b)} = a+b+c$.

18. Let $E$ denote the expression. Taking common denominators, we have

$$E = \frac{a(c-b)(x-b)(x-c)+b(a-c)(x-c)(x-a)+c(b-a)(x-a)(x-b)}{(x-a)(x-b)(x-c)(a-b)(b-c)(c-a)}.$$

The numerator vanishes when we put $a=b$. Hence it has $a-b$ as a factor. Similarly, it has the factors $b-c$ and $c-a$. It also vanishes when we put $x=0$. Hence $E=\frac{Mx}{(x-a)(x-b)(x-c)}$ for some constant $M$. Putting $a=0$, $b=1$ and $c=2$, we have $-\frac{1}{x-1}+\frac{1}{x-2}=\frac{M}{(x-1)(x-2)}$. Hence $M=1$ and $E=\frac{x}{(x-a)(x-b)(x-c)}$.

20. Let $E$ denote the expression. Taking common denominators, we have

$$\begin{aligned} E &= \frac{a^4(b-c)(b-d)(c-d)-b^4(a-c)(a-d)(c-d)}{(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)} \\ &\quad +\frac{c^4(a-b)(a-d)(b-d)+d^4(a-b)(a-c)(b-c)}{(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)}. \end{aligned}$$

The numerator vanishes when we put $a=b$. Hence it has $a-b$ as a factor. Similarly, it has the factors $a-c$, $a-d$, $b-c$, $b-d$ and $c-d$. Since it is of degree seven, the remaining factor must be $M(a+b+c+d)$. Putting $a=0$, $b=1$, $c=2$ and $d=3$, we find $M=6$. Hence $E=6(a+b+c+d)$.

22. We have $\sum(ab-c^2)(ca-b^2)=\sum b^2c^2+abc\sum a-\sum ab^2$. On the other hand, we have $(\sum bc)(\sum bc-\sum a^2)=\sum b^2c^2+2abc\sum a-abc\sum a-\sum ab^2$. Hence the two expressions are equal.

24. (a) Let $x=b-c$, $y=c-a$ and $z=a-b$. Then $x+y+z=0$. The desired sum is equal to $\sum x^3(y-z)=-(y-z)(z-x)(x-y)(x+y+z)=0$.

(b) Let $p = y - z$, $q = z - x$ and $r = x - y$. Then $p + q + r = 0$ and $q - r = x + y + z - 3x = -3x$. Hence the desired sum is equal to $\sum x(y-z)^3 = -\frac{1}{3}\sum p^3(q-r) = 0$.

26. Note that $x + y + z = a + b + c$ while $x - y = 2(b - a)$. We have

$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz &= \frac{1}{2}(x+y+z)((x-y)^2 + (y-z)^2 + (z-x)^2) \\ &= \frac{1}{2}(a+b+c)4((a-b)^2 + (b-c)^2 + (c-a)^2) \\ &= 4(a^3 + b^3 + c^3 - 3abc). \end{aligned}$$

28. The first term is equal to $\frac{p^4-(y^2+z^2)p^2s+y^2z^2s^2}{p^2yzs^2} = \frac{xp^2-x(y^2+z^2)s+(p^2+yz(y+z))s}{p^2s^2}$. Adding the three terms, the numerator simplifies to $(x+y+z)p^2+3p^2s = 4p^2s$. Hence the sum is $\frac{4}{s}$ as desired.

58. We have $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma) = q^2-2pr$.

60. We have

$$\begin{aligned} &\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma} \\ = &\frac{\alpha^2\gamma + \beta^2\gamma + \beta^2\alpha + \gamma^2\alpha + \gamma^2\beta + \alpha^2\beta}{\alpha\beta\gamma} \\ = &\frac{(\alpha+\beta+\gamma)(\alpha\beta+\beta\gamma+\gamma\alpha)}{\alpha\beta\gamma} - 3 \\ = &\frac{pq}{r} - 3. \end{aligned}$$

62. We have

$$\begin{aligned} &\alpha^2\beta\gamma + \alpha^2\beta\delta + \alpha^2\gamma\delta + \beta^2\gamma\delta + \beta^2\gamma\alpha + \beta^2\delta\alpha \\ &+\gamma^2\delta\alpha + \gamma^2\delta\beta + \gamma^2\alpha\beta + \delta^2\alpha\beta + \delta^2\alpha\gamma + \delta^2\beta\gamma \\ = &(\alpha+\beta+\gamma+\delta)(\alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta) - 4\alpha\beta\gamma\delta \\ = &(-p)(-r) - 4s \\ = &pr - 4s. \end{aligned}$$

## Answers to Exercises VIII

**1.** $-(b-c)(c-a)(a-b)(b+c)(c+a)(a+b)$.

**3.** $(b+c)(c+a)(a+b)$.

**5.** $3abc(b+c)(c+a)(a+b)$.

**7.** $80abc(a^2+b^2+c^2$.

**13.** 3.

**15.** $-1$.

**17.** $bc + ca + ab$.

**19.** $\frac{(p-x)(q-x)}{(a+x)(b+x)(c+x)}$.

**59.** $pq - r$.

**61.** $pq - 3r$.

**63.** $p^4 - 4p^2q + 2q^2 + 4pr - 4s$.